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Estimation of Lévy Processes

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Abstract

In this paper we investigate the estimation problems for the Lévy processes, for examples, compound Poisson process, stable processes and variance gamma processes. We employ the generalized method of moments and use the properties of the characteristic functions for the applications.

Key words: Lévy process, stable process, estimation of stochastic processes, generalized method of moments, characteristic function method.
JEL Classification: G12, G13

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1 Introduction

The Lévy processes are considered to be one of the most important stochastic models for the mathematical finance theory. For example the [Geometric Lévy Process & MEMM] pricing model is introduced as one of the pricing models for the incomplete market (see [9]). To apply these models to the empirical analysis, we have to estimate the Lévy processes from the given sequential data. So in this paper we investigate the estimation problems for the Lévy processes, for examples, compound Poisson process, stable processes and variance gamma processes, etc.

Two kinds of methods are available for the estimations: one is the maximum likelihood method, and the other is the generalized method of moments. We employ the generalized method of moments (or it should be called the method of characteristic functions). The reason is as follows. The Lévy processes and the corresponding infinitely divisible distributions are characterized by their generating triplets, and the generating triplets are explicitly contained in the characteristic functions. So it is natural for us to apply the generalized method of moments by the use of characteristic functions.

In Section 2 we explain the methods used in the following sections, and reduce the estimation problems for Lévy processes to these methods.

In the following sections we explain how to apply our methods to several examples.

2 Method of Moments

We first survey the method of moments.

2.1 Classical Method of Moments

If a distribution has moments of any degrees, then we can apply the classical method of moments. We first summarize that method. - 1) Moments:

Let Z be a random variable defined on some proper probability space. Then the k-th moment of Z (in the sense of distribution) is

$$m_k = E[Z^k]; \quad k = 0; 1; 2; \ldots \tag{2.1}$$

2) Sample moments:

Suppose that the time series data

$$\{z_j; j = 0; 1; \ldots; n\}; \quad z_0 = 0: \tag{2.2}$$

are given. Then the sample moments are

$$\hat{m}_k = \frac{1}{n}\sum_{j=1}^{n}(\xi_j)^k; \quad \xi_j = z_j - z_{j-1}; \quad j = 1; 2; \ldots n; \quad k = 1; 2; \ldots: \tag{2.3}$$

3) Classical Moment Equation:

The (classical) method of moments is such a method that the distribution is determined to to be one which satisfies the following classical moment equation

$$m_k = \hat{m}_k; \quad k = 1; 2; \ldots; N \tag{2.4}$$

where N is a number depending on the number of parameters.

4) Characteristic function:

The characteristic function $\phi(u)$ of Z is

$$\phi(u) = \phi_Z(u) = E[e^{iuZ}] = \exp(\psi(u)); \quad i = \sqrt{-1}: \tag{2.5}$$

We use the following notations

$$\phi^{(k)}(u) = \frac{d^k\phi}{du^k}(u); \quad \psi^{(k)}(u) = \frac{d^k\psi}{du^k}(u): \tag{2.6}$$

5) Formulae:

It is well-known that if Z has moments, then the following equalities hold.

$$m_k = E[Z^k] = \frac{1}{i^k}E[(iZ)^k] = \frac{1}{i^k}Á^{(k)}(0) \qquad (2.7)$$

So the classical moment equations are

$$\frac{1}{i^k}Á^{(k)}(0) = \hat{m}_k; \quad k = 1; 2; :::: \qquad (2.8)$$

Using the following formulae

$$Á^{(1)}(u) = Ã^{(1)}(u)Á(u) \qquad (2.9)$$

$$Á^{(2)}(u) = ³ Ã^{(2)}(u) + (Ã^{(1)}(u))^2 ´ Á(u) \qquad (2.10)$$

$$Á^{(3)}(u) = ³ Ã^{(3)}(u) + 3Ã^{(2)}(u)Ã^{(1)}(u) + (Ã^{(1)}(u))^3 ´ Á(u) \qquad (2.11)$$

$$Á^{(4)}(u) = ³ Ã^{(4)}(u) + 4Ã^{(3)}(u)Ã^{(1)}(u) + 3(Ã^{(2)}(u))^2 + 6Ã^{(2)}(u)(Ã^{(1)}(u))^2 + (Ã^{(1)}(u))^4 ´ Á(u) \qquad (2.12)$$

¢¢¢

we obtain the following results

$$m_1 = \frac{1}{i}Ã^{(1)}(0) \qquad (2.13)$$

$$m_2 = ¡ ³ Ã^{(2)}(0) + (Ã^{(1)}(0))^2 ´ \qquad (2.14)$$

$$m_3 = ¡\frac{1}{i^k} ³ Ã^{(3)}(0) + 3Ã^{(2)}(0)Ã^{(1)}(0) + (Ã^{(1)}(0))^3 ´ \qquad (2.15)$$

$$m_4 = Ã^{(4)}(0) + 4Ã^{(3)}(0)Ã^{(1)}(0) + 3(Ã^{(2)}(0))^2 + 6Ã^{(2)}(0)(Ã^{(1)}(0))^2 + (Ã^{(1)}(0))^4 \qquad (2.16)$$

¢¢¢

6) Transformation of the Moment Equation:

By the results obtained in 5), the moment equations (2.4) or (2.8) are transformed to

$$Ã^{(1)}(0) = im_1 \qquad (2.17)$$

$$Ã^{(2)}(0) = ¡ m_2 ¡ (Ã^{(1)}(0))^2$$

$$= ¡ m_2 ¡ (im_1)^2$$

$$
\begin{aligned}
&= -(m_2 - m_1^2) && (2.18)\\
\psi^{(3)}(0) &= -im_3 - \Big(3\psi^{(2)}(0)\psi^{(1)}(0) + (\psi^{(1)}(0))^3\Big) \\
&= -im_3 - 3(-m_2 + m_1^2)im_1 - (im_1)^3 \\
&= -i(m_3 - 3m_2m_1 + 2m_1^3) && (2.19)\\
\psi^{(4)}(0) &= m_4 - \Big(4\psi^{(3)}(0)\psi^{(1)}(0) + 3(\psi^{(2)}(0))^2 + 6\psi^{(2)}(0)(\psi^{(1)}(0))^2 + (\psi^{(1)}(0))^4\Big) \\
&= m_4 - 4(i(-m_3 + 3m_2m_1 - 2m_1^3)im_1) - 3(-m_2 + m_1^2)^2 \\
&\quad - 6(-m_2 + m_1^2)(im_1)^2 - (im_1)^4 \\
&= m_4 - 4m_3m_1 - 3m_2^2 + 12m_2m_1^2 - 6m_1^4, && (2.20)\\
&\cdots
\end{aligned}
$$

Set

$$
\begin{aligned}
\hat{\mu}_1 &= \hat{m}_1 && (2.21)\\
\hat{\mu}_2 &= \hat{m}_2 - \hat{m}_1^2 && (2.22)\\
\hat{\mu}_3 &= \hat{m}_3 - 3\hat{m}_2\hat{m}_1 + 2\hat{m}_1^3 && (2.23)\\
\hat{\mu}_4 &= \hat{m}_4 - 4\hat{m}_3\hat{m}_1 - 3\hat{m}_2^2 + 12\hat{m}_2\hat{m}_1^2 - 6\hat{m}_1^4, && (2.24)\\
&\cdots
\end{aligned}
$$

then the moment equations are equivalent to the following equations.

$$
\begin{aligned}
\psi^{(1)}(0) &= i\hat{\mu}_1 && (2.25)\\
\psi^{(2)}(0) &= -\hat{\mu}_2 && (2.26)\\
\psi^{(3)}(0) &= -i\hat{\mu}_3 && (2.27)\\
\psi^{(4)}(0) &= \hat{\mu}_4, && (2.28)\\
&\cdots
\end{aligned}
$$

2.2 Generalized Method of Moments (Characteristic Function Methods)

If a distribution does not have moments, then we can not apply the classical method of moments. However the generalized method of moments can be applied to such cases.

1) Sample Characteristic Function:

The characteristic function (in the sense of distribution) of Z is already defined. We here introduce the sample characteristic function by the following formula

$$\hat{\phi}(u) = \frac{1}{n}\sum_{j=1}^{n} e^{iu\xi_j}; \quad \xi_j = z_j - z_{j-1}. \tag{2.29}$$

Note that $\hat{\phi}(u)$ is a consistent estimator of $\phi(u)$:

$$\lim_{n\to\infty} \hat{\phi}(u) = \phi(u); \quad -\infty < u < \infty. \tag{2.30}$$

2) Generalized Moment Equation:

The following equation is called the generalized moment equation.

$$\phi(u) = \hat{\phi}(u); \quad -\infty < u < \infty. \tag{2.31}$$

2.3 Estimation of Lévy Processes

The Lévy process Z_t is characterized by the generating triplet $(\sigma^2; \nu(dx); b)$ (or $(\sigma^2; \nu(dx); b_c)_c$). Set $Z = Z_1$. It is well-known that the distribution of Z is an infinitely divisible distribution, and that the corresponding characteristic function $\phi(u)$ is

$$\begin{aligned} \phi(u) &= E[e^{iuZ}] = \exp(\psi(u)) \\ &= \exp\left\{ -\frac{\sigma^2}{2}u^2 + \int_{|x|<1}(e^{iux} - 1 - iux)\nu(dx) \right. \\ &\qquad \left. + \int_{|x|\geq 1}(e^{iux} - 1)\nu(dx) + ibu \right\} \qquad (2.32) \\ &= \exp\left\{ -\frac{\sigma^2}{2}u^2 + \int_{(-\infty;\infty)}\left(e^{iux} - 1 - iuxc(x)\right)\nu(dx) + ib_c u \right\} \qquad (2.33) \end{aligned}$$

What we have to do is to estimate the generating triplet $(\sigma^2; \nu(dx); b)$ (or $(\sigma^2; \nu(dx); b_c)_c$) of the distribution of Z. Set $\{\tilde{Z}_k = Z_k - Z_{k-1}; k = 1, 2, \ldots\}$, then $\{\tilde{Z}_k; k = 1, 2, \ldots\}$ is i.i.d. with the same distribution as Z since Lévy process has temporally homogeneous independent increment. So, if we are given an sequential data of an Lévy process, then we can apply the method described above to estimate the distribution of Z.

3 Compound Poisson Model with Normal Lévy Measure

In this section we investigate the estimation problem of compound Poisson processes. For the simplicity and the usefulness, we consider only the case of normal Lévy Measure. (Other cases can be treated in the same manner.) So we suppose that the Lévy Measure is

$$\nu(dx) = cg(x;m;v)dx = c\frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{(x-m)^2}{2v}\right)dx \qquad (3.1)$$

and the characteristic function $\phi(u)$ is of the following form with the parameters $(c;m;v;b_0)$,

$$\phi(u) = E[e^{iuZ}] = \exp(\psi(u)) \qquad (3.2)$$

$$\psi(u) = ib_0u + \int_{-\infty}^{\infty}(e^{iux} - 1)\nu(dx) = ib_0u + c(\hat{g}(u) - 1); \qquad (3.3)$$

where

$$\hat{g}(u) = \int_{-\infty}^{\infty} e^{iux}g(x;m;v)dx = \exp\left(imu - \frac{1}{2}vu^2\right): \qquad (3.4)$$

We can apply the classical method of moments described in §2.1 to estimate the parameters $(c;m;v;b_0)$ as follows.

$$\psi^{(1)}(u) = ib_0 + c(im - vu)\hat{g}(u) \qquad (3.5)$$

$$\psi^{(2)}(u) = \left(-cv + c(im - vu)^2\right)\hat{g}(u) \qquad (3.6)$$

$$\psi^{(3)}(u) = \left(-3cv(im - vu) + c(im - vu)^3\right)\hat{g}(u) \qquad (3.7)$$

$$\psi^{(4)}(u) = \left(3cv^2 - 6cv(im - vu)^2 + c(im - vu)^4\right)\hat{g}(u) \qquad (3.8)$$

$$\psi^{(1)}(0) = i(b_0 + cm) \qquad (3.9)$$

$$\psi^{(2)}(0) = -c(v + m^2) \qquad (3.10)$$

$$\psi^{(3)}(0) = -ic(v + m^2)m \qquad (3.11)$$

$$\psi^{(4)}(0) = c(3v^2 + 6vm^2 + m^4) \qquad (3.12)$$

Thus the classical moment equations (2.8) are

$$b_0 + cm = \hat{h}_1 \tag{3.13}$$
$$c(v + m^2) = \hat{h}_2 \tag{3.14}$$
$$c(v + m^2)m = \hat{h}_3 \tag{3.15}$$
$$c(3v^2 + 6vm^2 + m^4) = \hat{h}_4. \tag{3.16}$$

This equations are solved in the following way. From (3.14) and (3.15)

$$m = \frac{\hat{h}_3}{\hat{h}_2}. \tag{3.17}$$

By (3.14)

$$v + m^2 = \frac{\hat{h}_2}{c}; \tag{3.18}$$

and (3.16) is

$$3c(v + m^2)^2 - 2cm^4 = \hat{h}_4. \tag{3.19}$$

From these three equalities we obtain

$$3\frac{\hat{h}_2^2}{c} - 2c\left(\frac{\hat{h}_3}{\hat{h}_2}\right)^4 = \hat{h}_4; \tag{3.20}$$

Solving this equation, we obtain (remark that $c > 0$)

$$c = \frac{\sqrt{\hat{h}_4^2 + 24\frac{\hat{h}_3^4}{\hat{h}_2^2}} - \hat{h}_4}{4\left(\frac{\hat{h}_3}{\hat{h}_2}\right)^4} \tag{3.21}$$

and

$$b_0 = \hat{h}_1 - cm \tag{3.22}$$
$$v = \frac{\hat{h}_2}{c} - m^2 \tag{3.23}$$

Summarizing the above results, we have obtained

Result:

$$\hat{m} = \frac{\hat{h}_3}{\hat{h}_2} \tag{3.24}$$

$$\hat{c} = \frac{\sqrt{\hat{h}_4^2 + 24\frac{\hat{h}_3^4}{\hat{h}_2^2}} - \hat{h}_4}{4\left(\frac{\hat{h}_3}{\hat{h}_2}\right)^2} \tag{3.25}$$

$$\hat{b}_0 = \hat{h}_1 - \hat{c}\hat{m} \tag{3.26}$$

$$\hat{v} = \frac{\hat{h}_2}{\hat{c}} - \hat{m}^2 \tag{3.27}$$

4 Jump Diffusion (Compound Poisson Diffusion) Model

Suppose that the Lévy process Z_t is $Z_t = \sigma W_t + b_0 t + J_t$, where J_t is a compound Poisson process. Then the generating triplet of Z_t is $(\sigma^2; \nu(dx); b_0)_0$, and the Lévy measure $\nu(dx)$ is

$$\nu(dx) = c\rho(dx) \tag{4.1}$$

where c is a positive constant and $\rho(dx)$ is a probability on $(-\infty, \infty)$ such that $\rho(\{0\}) = 0$. Then the characteristic function $\phi(u)$ is of the following form

$$\phi(u) = E[e^{iuZ}] = \exp(\psi(u)) \tag{4.2}$$

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + ib_0 u + c(\hat{\rho}(u) - 1), \tag{4.3}$$

where

$$\hat{\rho}(u) = \int_{-\infty}^{\infty} e^{iux}\rho(dx). \tag{4.4}$$

We mention here that $|\hat{\rho}(u)| \leq 1$, so

$$|c(\hat{\rho}(u) - 1)| \leq 2c. \tag{4.5}$$

From this we obtain

Proposition 1

$$-2 \lim_{|u| \to \infty} \frac{\mathrm{Re}[\psi(u)]}{u^2} = \sigma^2. \tag{4.6}$$

Example 1 (Discrete Lévy measure) The parameter σ^2 is estimated from the sample characteristic function by the use of the above proposition.

Suppose that the Lévy measure $\nu(dx)$ is discrete, namely in the following form

$$\nu(dx) = c\rho(dx) = c\sum_{j=1}^{d} p_j \delta_{a_j}(dx); \quad p_j \geq 0; j = 1, 2, \ldots, d; \quad \sum_{j=1}^{d} p_j = 1. \tag{4.7}$$

Then

$$\psi(u) = -\frac{1}{2}\sigma^2 u^2 + ib_0 u + c\sum_{j=1}^{d} p_j(e^{iua_j} - 1); \tag{4.8}$$

The estimations of the parameters b_0, c and $p_j; j = 1, \ldots, d$ can be done by the use of the $(d+1)$ moment equations given in §2.1.

Example 2 (Normal L¶evy Measure) The estimations of ¾^2 and b_0 are the same as the above example. When we have obtained the values of these parameters, the rest part is carried on in the same way as in x3.

5 Stable Process

$S_\alpha(\beta;\tau;c)$, $(0 < \alpha < 2,\ -1 \le \beta \le 1,\ -\infty < \tau < \infty,\ c > 0)$.

In [11] the moment estimation is given. We have modified his methods.
Characteristic Function:

$$\phi(u) = \phi_{\text{stable}}(u) = \exp(\psi(u)) \tag{5.1}$$

$$\psi(u) = \begin{cases} -c|u|^\alpha\left(1 - i\beta\tan\frac{\pi\alpha}{2}\operatorname{sgn}(u)\right) + i\tau u; & \text{for } \alpha \neq 1 \\ -c|u|\left(1 + i\beta\frac{2}{\pi}\operatorname{sgn}(u)\log|u|\right) + i\tau u; & \text{for } \alpha = 1. \end{cases} \tag{5.2}$$

Polar coordinate:

$$\phi(u) = \rho(u)e^{i\theta(u)} \tag{5.3}$$

where

$$\rho(u) = |\phi(u)| = e^{-c|u|^\alpha} \tag{5.4}$$

$$\theta(u) = \operatorname{Im}[\psi(u)] = \begin{cases} c|u|^\alpha\beta\tan\frac{\pi\alpha}{2}\operatorname{sgn}(u) + \tau u & \text{for } \alpha \neq 1 \\ -c|u|\beta\frac{2}{\pi}\operatorname{sgn}(u)\log|u| + \tau u & \text{for } \alpha = 1 \end{cases} \tag{5.5}$$

Judgement of $\alpha = 1$ or $\alpha \neq 1$:

The following formulae

$$\log|\phi(u)| = -c|u|^\alpha \tag{5.6}$$

and

$$\lim_{u\to\infty}\frac{\log|\log|\phi(u)||}{\log|u|} = \alpha \tag{5.7}$$

follow from (5.3). Using these formulae, we can judge whether $\alpha = 1$ or $\alpha \neq 1$, depending on the fact that the function

$$y = \log|\hat{\phi}(u)| \tag{5.8}$$

is almost linear or not.

(i) Case of $\alpha \neq 1$.

Assume that we have judged that $\alpha \neq 1$. Choose u_1 and u_2 such that $u_1, u_2 \neq 0$ and $u_1 \neq u_2$, and we use the moment equation

$$|\phi(u)| = |\hat{\phi}(u)|; \quad \text{for} \quad u = u_1, u_2 \tag{5.9}$$

which follows from the generalized moment equation (2.31). This is equivalent to

$$c|u_1|^{\alpha} = -\log|\hat{\phi}(u_1)|; \qquad c|u_2|^{\alpha} = -\log|\hat{\phi}(u_2)| \tag{5.10}$$

Remark 1 Since $|\hat{\phi}(u)| \leq 1$, it holds true that $-\log|\hat{\phi}(u)| \geq 0$.

Solving this equation, we obtain the following estimators

$$\hat{\alpha} = \frac{\log\left(\frac{\log|\hat{\phi}(u_1)|}{\log|\hat{\phi}(u_2)|}\right)}{\log|\frac{u_1}{u_2}|} \tag{5.11}$$

$$\hat{c} = -\frac{\log|\hat{\phi}(u_1)|}{|u_1|^{\hat{\alpha}}} \tag{5.12}$$

For the estimation of β and τ, we use the polar expression (5.3). Set

$$\hat{\theta}(u) = \operatorname{Im}\hat{\Phi}(u) = \operatorname{Im}(\operatorname{Log}\hat{\phi}(u)) \tag{5.13}$$

and choose u_3 and u_4 such that $u_3, u_4 \neq 0$, $u_3 \neq u_4$ and small enough. Using the moment equation $\theta(u) = \hat{\theta}(u)$, which follows from the generalized moment equation (2.31), we obtain the equations

$$\hat{c}|u_3|^{\hat{\alpha}}\beta \tan\frac{\pi\hat{\alpha}}{2}\operatorname{sgn}(u_3) + \tau u_3 = \hat{\theta}(u_3) \tag{5.14}$$

$$\hat{c}|u_4|^{\hat{\alpha}}\beta \tan\frac{\pi\hat{\alpha}}{2}\operatorname{sgn}(u_4) + \tau u_4 = \hat{\theta}(u_4) \tag{5.15}$$

These are linear equations for β and τ, and the solution is

$$\begin{aligned} \hat{\beta} &= \frac{\hat{\theta}(u_3)u_4 - \hat{\theta}(u_4)u_3}{\hat{c}(\tan\frac{\pi\hat{\alpha}}{2})\,(|u_3|^{\hat{\alpha}}\operatorname{sgn}(u_3)u_4 - |u_4|^{\hat{\alpha}}\operatorname{sgn}(u_4)u_3)} \\ &= \frac{\hat{\theta}(u_3)u_4 - \hat{\theta}(u_4)u_3}{\hat{c}(\tan\frac{\pi\hat{\alpha}}{2})(|u_3|^{\hat{\alpha}-1} - |u_4|^{\hat{\alpha}-1})u_3u_4}, \end{aligned} \tag{5.16}$$

$$\begin{aligned} \hat{\tau} &= \frac{|u_3|^{\hat{\alpha}}\operatorname{sgn}(u_3)\hat{\theta}(u_4) - |u_4|^{\hat{\alpha}}\operatorname{sgn}(u_4)\hat{\theta}(u_4)}{|u_3|^{\hat{\alpha}}\operatorname{sgn}(u_3)u_4 - |u_4|^{\hat{\alpha}}\operatorname{sgn}(u_4)u_3} \\ &= \frac{|u_3|^{\hat{\alpha}-1}u_3\hat{\theta}(u_4) - |u_4|^{\hat{\alpha}-1}u_4\hat{\theta}(u_4)}{(|u_3|^{\hat{\alpha}-1} - |u_4|^{\hat{\alpha}-1})u_3u_4}. \end{aligned} \tag{5.17}$$

(ii) Case of $\alpha = 1$.

Assume that we have judged that $\alpha = 1$. Then $\frac{\log|\phi(u)|}{|u|} = \frac{\log \rho(u)}{|u|} = -c$. Choose $u_1 \neq 0$. The estimator of c is

$$\hat{c} = -\frac{\log|\phi(u_1)|}{|u_1|}. \tag{5.18}$$

The equations for β and τ are followed from the moment equation $\mu(u) = \hat{\mu}(u)$ with $\alpha = 1$ and $\hat{c}$ of (5.18), so

$$\left(-\hat{c}\beta\frac{2}{\pi}\log|u_3| + \tau\right)u_3 = \hat{\mu}(u_3) \tag{5.19}$$

$$\left(-\hat{c}\beta\frac{2}{\pi}\log|u_4| + \tau\right)u_4 = \hat{\mu}(u_4) \tag{5.20}$$

The solution is

$$\hat{\beta} = \frac{\hat{\mu}(u_3)u_4 - \hat{\mu}(u_4)u_3}{-\hat{c}\frac{2}{\pi}(\log|u_3| - \log|u_4|)u_3u_4} \tag{5.21}$$

$$\hat{\tau} = \frac{(\log|u_3|)u_3\hat{\mu}(u_4) - (\log|u_4|)u_4\hat{\mu}(u_3)}{(\log|u_3| - \log|u_4|)u_3u_4} \tag{5.22}$$

6 Variance Gamma Model

$VG(C; c_1; c_2; b_0)$

Lévy measure, generating triplets, and characteristic function ($\phi(u)$):
The Lévy measure is

$$\nu(dx) = C\left(1_{\{x<0\}}\exp(-c_1|x|) + 1_{\{x>0\}}\exp(-c_2|x|)\right)|x|^{-1}dx; \tag{6.1}$$

where $C; c_1; c_2$ are positive constants.

The generating triplet is $(0; \nu(dx); b_0)_0$, and the characteristic function $\phi_{VG}(u)$ is

$$\phi_{VG}(u) = \exp\left\{ i b_0 u - C\left(\log(1+\frac{iu}{c_1}) + \log(1-\frac{iu}{c_2})\right)\right\} \tag{6.2}$$

$$= e^{ib_0u}\left(\frac{1}{\left(1+\frac{iu}{c_1}\right)\left(1-\frac{iu}{c_2}\right)}\right)^C \tag{6.3}$$

6.1 Estimation by Classical Method of Moments

Set

$$\phi_{VG}(u) = \exp\{\psi(u)\}$$
$$\psi(u) = i b_0 u - C\left(\log(1+\frac{iu}{c_1}) + \log(1-\frac{iu}{c_2})\right); \tag{6.4}$$

then

$$\psi^{(1)}(u) = i b_0 + iC\left(\frac{1}{c_2 - iu} - \frac{1}{c_1 - iu}\right); \tag{6.5}$$
$$\psi^{(2)}(u) = -C\left(\frac{1}{(c_2 - iu)^2} + \frac{1}{(c_1 - iu)^2}\right) \tag{6.6}$$
$$\psi^{(3)}(u) = -2iC\left(\frac{1}{(c_2 - iu)^3} - \frac{1}{(c_1 - iu)^3}\right) \tag{6.7}$$
$$\psi^{(4)}(u) = 6C\left(\frac{1}{(c_2 - iu)^4} + \frac{1}{(c_1 - iu)^4}\right): \tag{6.8}$$

So we obtain

$$\psi^{(1)}(0) = i\left(b_0 + C\left(\frac{1}{c_2} - \frac{1}{c_1}\right)\right); \tag{6.9}$$
$$\psi^{(2)}(0) = -C\left(\frac{1}{c_2^2} + \frac{1}{c_1^2}\right) \tag{6.10}$$

$$\psi^{(3)}(0) = -2C\left(\frac{1}{c_2^3} - \frac{1}{c_1^3}\right) \tag{6.11}$$

$$\psi^{(4)}(0) = 6C\left(\frac{1}{c_2^4} + \frac{1}{c_1^4}\right). \tag{6.12}$$

From the above results, the moment equations (2.8) are

$$b_0 + C\left(\frac{1}{c_2} - \frac{1}{c_1}\right) = \hat{h}_1 \tag{6.13}$$

$$C\left(\frac{1}{c_2^2} + \frac{1}{c_1^2}\right) = \hat{h}_2 \tag{6.14}$$

$$2C\left(\frac{1}{c_2^3} - \frac{1}{c_1^3}\right) = \hat{h}_3 \tag{6.15}$$

$$6C\left(\frac{1}{c_2^4} + \frac{1}{c_1^4}\right) = \hat{h}_4. \tag{6.16}$$

We solve these equations as follows. Set

$$x_1 = -\frac{1}{c_1} < 0; \quad x_2 = \frac{1}{c_2} > 0; \tag{6.17}$$

then

$$x_1 + x_2 = \frac{\hat{h}_1 - b_0}{C} \tag{6.18}$$

$$x_1^2 + x_2^2 = \frac{\hat{h}_2}{C} \tag{6.19}$$

$$x_1^3 + x_2^3 = \frac{\hat{h}_3}{C} \tag{6.20}$$

$$x_1^4 + x_2^4 = \frac{\hat{h}_4}{C} \tag{6.21}$$

By (6.19),

$$(x_1 + x_2)^2 = x_1^2 + x_2^2 + 2x_1x_2 = \frac{\hat{h}_2}{C} + 2x_1x_2; \tag{6.22}$$

and from (6.18), (6.22)

$$x_1x_2 = \frac{1}{2}\left(\frac{(\hat{h}_1 - b_0)^2}{C^2} - \frac{\hat{h}_2}{C}\right) \tag{6.23}$$

When b_0 and C are given, by (6.18) and (6.23), x_1 and x_2 are the solutions of the following equation

$$x^2 - \frac{(\hat{h}_1 - b_0)}{C} + \frac{1}{2}\left(\frac{(\hat{h}_1 - b_0)^2}{C^2} - \frac{\hat{h}_2}{C}\right) = 0. \tag{6.24}$$

Next we derive the equations for C and b_0. Using (6.18) and (6.19), we obtain

$$
\begin{aligned}
(x_1 + x_2)^3 &= (x_1 + x_2)(x_1^2 - x_1 x_2 + x_2^2) \\
&= \frac{(\hat{h}_1 - b_0)}{C}\left(\frac{\hat{h}_2}{C} - \frac{1}{2}\left(\frac{(\hat{h}_1 - b_0)^2}{C^2} - \frac{\hat{h}_2}{C}\right)\right) \\
&= \frac{(\hat{h}_1 - b_0)}{C}\left(\frac{3}{2}\frac{\hat{h}_2}{C} - \frac{1}{2}\frac{(\hat{h}_1 - b_0)^2}{C^2}\right) \qquad (6.25)
\end{aligned}
$$

So, by (6.20),

$$
\hat{h}_3 = (\hat{h}_1 - b_0)\left(\frac{3}{2}\frac{\hat{h}_2}{C} - \frac{1}{2}\frac{(\hat{h}_1 - b_0)^2}{C^2}\right) \qquad (6.26)
$$

and

$$
\hat{h}_3 C^2 - 3\hat{h}_2(\hat{h}_1 - b_0)C + (\hat{h}_1 - b_0)^3 = 0. \qquad (6.27)
$$

Using (6.19) and (6.23), we obtain

$$
\begin{aligned}
x_1^4 + x_2^4 &= (x_1^2 + x_2^2)^2 - 2x_1^2 x_2^2 \\
&= \left(\frac{\hat{h}_2}{C}\right)^2 - 2\left(\frac{1}{2}\left(\frac{(\hat{h}_1 - b_0)^2}{C^2} - \frac{\hat{h}_2}{C}\right)\right)^2 \\
&= \frac{1}{2}\frac{\hat{h}_2^2}{C^2} + \frac{\hat{h}_2(\hat{h}_1 - b_0)^2}{C^3} - \frac{1}{2}\frac{(\hat{h}_1 - b_0)^4}{C^4}. \qquad (6.28)
\end{aligned}
$$

By (6.21),

$$
\hat{h}_4 = \frac{1}{2}\frac{\hat{h}_2^2}{C} + \frac{\hat{h}_2(\hat{h}_1 - b_0)^2}{C^2} - \frac{1}{2}\frac{(\hat{h}_1 - b_0)^4}{C^3}. \qquad (6.29)
$$

and so

$$
\hat{h}_4 C^3 - \frac{1}{2}\hat{h}_2^2 C^2 - \hat{h}_2(\hat{h}_1 - b_0)^2 C + \frac{1}{2}(\hat{h}_1 - b_0)^4 = 0. \qquad (6.30)
$$

By $(6.30) - (6.27) \times (\hat{h}_1 - b_0)$, we obtain

$$
\hat{h}_2(\hat{h}_1 - b_0)^2 - 2\hat{h}_3 C(\hat{h}_1 - b_0) - \hat{h}_2^2 C + 2\hat{h}_4 C^2 = 0. \qquad (6.31)
$$

Eliminating $(\hat{h}_1 - b_0)$ by $(6.27) \times \hat{h}_2 - (6.31) \times (\hat{h}_1 - b_0)$, we obtain the following equation

$$
\hat{h}_3 C(\hat{h}_1 - b_0)^2 - (\hat{h}_2^2 + \hat{h}_4 C)(\hat{h}_1 - b_0) + \hat{h}_2\hat{h}_3 C = 0, \qquad (6.32)
$$

namely

$$
(\hat{h}_2\hat{h}_3 - \hat{h}_4(\hat{h}_1 - b_0))C + \hat{h}_3(\hat{h}_1 - b_0)^2 - \hat{h}_2^2(\hat{h}_1 - b_0) = 0. \qquad (6.33)
$$

Solving this equation, we get the solution

$$C = \frac{\hat{a}_3(\hat{a}_1 - b_0)^2 - \hat{a}_2^2(\hat{a}_1 - b_0)}{\hat{a}_4(\hat{a}_1 - b_0) - \hat{a}_2\hat{a}_3} = \frac{\hat{a}_3Y^2 - \hat{a}_2^2Y}{\hat{a}_4Y - \hat{a}_2\hat{a}_3}; \qquad (6.34)$$

where we set $Y = (\hat{a}_1 - b_0)$.

The equation (6.31) is writen as

$$\hat{a}_2Y^2 - 2\hat{a}_3CY - \hat{a}_2^2C + 2\hat{a}_4C^2 = 0. \qquad (6.35)$$

Using (6.34), we obtain

$$\begin{aligned} &\hat{a}_2(\hat{a}_4Y - \hat{a}_2\hat{a}_3)^2Y^2 - 2\hat{a}_3(\hat{a}_3Y^2 - \hat{a}_2^2Y)(\hat{a}_4Y - \hat{a}_2\hat{a}_3)Y \\ &- \hat{a}_2^2(\hat{a}_3Y^2 - \hat{a}_2^2Y)(\hat{a}_4Y - \hat{a}_2\hat{a}_3)C + 2\hat{a}_4(\hat{a}_3Y^2 - \hat{a}_2^2Y)^2 = 0. \end{aligned} \qquad (6.36)$$

namely

$$\hat{a}_2^3\hat{a}_3^2Y^4 + (2\hat{a}_2\hat{a}_3^3 - 2\hat{a}_2^2\hat{a}_3 - 3\hat{a}_2^2\hat{a}_3\hat{a}_4)Y^3 + 3\hat{a}_2^4\hat{a}_4Y^2 - \hat{a}_2^5\hat{a}_3Y = 0 \qquad (6.37)$$

We know from (6.34) that $Y \neq 0$ because $C > 0$. Therefore we obtain

$$\hat{a}_2^3\hat{a}_3^2Y^3 + (2\hat{a}_2\hat{a}_3^3 - 2\hat{a}_2^2\hat{a}_3 - 3\hat{a}_2^2\hat{a}_3\hat{a}_4)Y^2 + 3\hat{a}_2^4\hat{a}_4Y - \hat{a}_2^5\hat{a}_3 = 0. \qquad (6.38)$$

Suppose that the above equation (6.38) has a solution. Using this solution, we obtain the following procedure.

Result:

Let Y be a solution of (6.38). Then

$$\hat{b}_0 = \hat{a}_1 - Y \qquad (6.39)$$

$$\hat{C} = \frac{\hat{a}_3Y^2 - \hat{a}_2^2Y}{\hat{a}_4Y - \hat{a}_2\hat{a}_3}; \qquad (6.40)$$

where we assume that $\hat{C}$ is positive. Next let $(\hat{x}_1, \hat{x}_2)$ be the solution of (6.24), and suppose that $\hat{x}_1 < 0 < \hat{x}_2$. Then

$$\hat{c}_1 = -\frac{1}{x_1} \qquad (6.41)$$

$$\hat{c}_2 = \frac{1}{x_2} \qquad (6.42)$$

6.2 Estimation by Generilized method of Moments

We can apply the generalized method of moment to the Variance Gamma model. The characteristic function $\phi_{VG}(u)$ is given in (6.2) and (VG.chf.2).

Estimation of C:

It is easy to see that

$$\lim_{u \to \infty} \frac{\mathrm{Re}[\log \phi(u)]}{\log u} = -2C \tag{6.43}$$

Using this formula, we obtain an estimator $\hat{C}$ of C

$$\hat{C} = -\frac{1}{2}\frac{\mathrm{Re}[\log \hat{\phi}(u_1)]}{u_1}; \tag{6.44}$$

for some large number u_1.

Estimation of b_0, c_1 and c_2:

Let $\hat{C}$ be the estimator of C. The moment equations (2.8) are in this case

$$b_0 + \hat{C}\left(\frac{1}{c_2} - \frac{1}{c_1}\right) = \hat{h}_1 \tag{6.45}$$

$$\hat{C}\left(\frac{1}{c_2^2} + \frac{1}{c_1^2}\right) = \hat{h}_2 \tag{6.46}$$

$$2\hat{C}\left(\frac{1}{c_2^3} - \frac{1}{c_1^3}\right) = \hat{h}_3: \tag{6.47}$$

We can solve this equation for b_0, c_1 and c_2.

7 CGMY Process

$CGMY(C, G, M, Y, b_1)$

7.1 Characteristic Function

Lévy measure, generating triplets:

The Lévy measure of the CGMY process is

$$\nu(dx) = C\left(1_{\{x<0\}}\exp(-G|x|) + 1_{\{x>0\}}\exp(-M|x|)\right)|x|^{-(1+Y)}dx, \quad (7.1)$$

where $C > 0, G \geq 0, M \geq 0, Y < 2$. If $Y \leq 0$, then $G > 0$ and $M > 0$ are assumed. We mention here that the case $Y = 0$ is the VG process case, and the case $G = M = 0$ and $0 < Y < 2$ is the symmetric stable process case. In the sequel we assume that $G, M > 0$.

Since

$$\int_{|x|\geq 1} |x|\nu(dx) < \infty \quad (7.2)$$

we can adopt the generating triplet $(0, \nu, b_1)_1$. And we know that b_1 is the mean of the distribution [1].

If $Y < 1$, then the following condition

$$\int_{|x|<1} |x|\nu(dx) < \infty \quad (7.3)$$

is satisfied. So in this case we have another expression of the generating triplet $(0, \nu, b_0)_0$.

The characteristic function ($\phi_{CGMY}(u)$):

(1) The case $Y = 0$.

This case is Variance Gamma case. The characteristic function $\phi_{VG}(u)$ is

$$\begin{aligned}
\phi_{VG}(u) &= \exp\left\{ i b_0 u + \int_{|x|>0} (e^{iux} - 1)\nu(dx) \right\} \\
&= \exp\left\{ i b_0 u - C\left(\log(1 + \frac{iu}{G}) + \log(1 - \frac{iu}{M})\right) \right\} \\
&= e^{i b_0 u}\left(\frac{1}{\left(1 + \frac{iu}{G}\right)\left(1 - \frac{iu}{M}\right)} \right)^C \quad (7.4)
\end{aligned}$$

Remark 2 If we adopt the generating triplet $(0;\nu;b_1)_1$, then

$$b_1 = b_0 + \int_{-\infty}^{\infty} x\nu(dx) = b_0 + C\left(\frac{1}{M} - \frac{1}{G}\right) \tag{7.5}$$

and

$$\begin{aligned}
\phi_{VG}(u) &= \exp\left\{ i\left(b_1 - C\left(\frac{1}{M} - \frac{1}{G}\right)\right)u - C\left(\log(1+\frac{iu}{G}) + \log(1-\frac{iu}{M})\right)\right\} \\
&= e^{i\left(b_1 - C\left(\frac{1}{M} - \frac{1}{G}\right)\right)u}\left(\frac{1}{\left(1+\frac{iu}{G}\right)\left(1-\frac{iu}{M}\right)}\right)^{C}
\end{aligned} \tag{7.6}$$

(2) The case $Y < 1$ and $Y \neq 0$.

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_0u + \int_{|x|>0}(e^{iux} - 1)\nu(dx)\right\} \\
&= \exp\left\{ ib_0u + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\right\}
\end{aligned} \tag{7.7}$$

If we adopt another expression of the generating triplet, $(0;\nu(dx);b_1)_1$, then we obtain

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_1u + \int_{|x|>0}(e^{iux} - 1 - iux)\nu(dx)\right\} \\
&= \exp\Big\{ i\left(b_1 + C\Gamma(-Y)Y(M^{Y-1} - G^{Y-1})\right)u \\
&\qquad + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\Big\}
\end{aligned} \tag{7.8}$$

Remark 3 (i) It follows from these formulae that

$$b_0 = b_1 + C\Gamma(-Y)Y(M^{Y-1} - G^{Y-1}). \tag{7.9}$$

(ii) The right hand side of (A.7) or (A.8) converged to the characteristic function of VG when $Y \to 0$, namely

$$\begin{aligned}
&\lim_{Y\to 0}\left\{ ib_0u - C\frac{\Gamma(1-Y)}{Y}\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\right\} \\
&= ib_0u - C\left(\log(M - iu) - \log M + \log(G + iu) - \log G\right) \\
&= ib_0u - C\left(\log(1 - \frac{iu}{M}) + \log(1 + \frac{iu}{G})\right).
\end{aligned} \tag{7.10}$$

(3) The case $Y = 1$.

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_1 u + \int_{|x|>0} (e^{iux} - 1 - iux)\nu(dx) \right\} \\
&= \exp\left\{ ib_1 u + C\left(\left((M - iu)\log(1 - \frac{iu}{M}) + iu\right) + \left((G + iu)\log(1 + \frac{iu}{G}) - iu\right)\right)\right\} \\
&= \exp\left\{ ib_1 u + C\left((M - iu)\log(1 - \frac{iu}{M}) + (G + iu)\log(1 + \frac{iu}{G})\right)\right\} \qquad (7.11)
\end{aligned}$$

(4) The case $1 < Y < 2$.

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_1 u + \int_{|x|>0} (e^{iux} - 1 - iux)\nu(dx) \right\} \\
&= \exp\Big\{ ib_1 u + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right) \\
&\quad + iuC\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\Big\} \\
&= \exp\Big\{ i\left(b_1 + C\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\right)u \\
&\quad + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\Big\} \qquad (7.12)
\end{aligned}$$

Remark 4 The last formula is the same with the case (2).

Remark 5 The right hand side of (A.13) converges to the right hand side of (A.11), namely

$$\begin{aligned}
&\lim_{Y \to 1} \Big\{ i\left(b_1 + C\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\right)u \\
&\quad + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\Big\} \\
= &\lim_{Y \to 1} \Big\{ i\left(b_1 + C\frac{\Gamma(2 - Y)}{Y - 1}\left(M^{Y-1} - G^{Y-1}\right)\right)u \\
&\quad + C\frac{\Gamma(2 - Y)}{Y(Y - 1)}\Big((M - iu)((M - iu)^{Y-1} - 1) - M(M^{Y-1} - 1) \\
&\quad + (G + iu)((G + iu)^{Y-1} - 1) - G(G^{Y-1} - 1)\Big)\Big\} \\
= &\ i(b_1 + C(\log M - \log G))u \\
&\quad + C((M - iu)\log(M - iu) - M\log M + (G + iu)\log(G + iu) - G\log G) \\
= &\ ib_1 u + C\left((M - iu)\log(1 - \frac{iu}{M}) + (G + iu)\log(1 + \frac{iu}{G})\right) \qquad (7.13)
\end{aligned}$$

7.2 Properties of Characteristic Function

Proposition 2

$$\lim_{u\to\infty}\frac{\log|\mathrm{Re}[\log\phi(u)]|}{\log u}=\begin{cases}Y; & Y>0;\\ 0; & Y\le 0\end{cases}\tag{7.14}$$

Proof (see Appendix B)

Proposition 3 (i) If $0<Y<2$

$$\lim_{u\to\infty}\frac{\mathrm{Re}[\log\phi(u)]}{u^Y}=\begin{cases}2C\Gamma(-Y)\cos\left(\frac{\pi}{2}Y\right); & Y\neq 1\\ -\pi C; & Y=1\end{cases}\tag{7.15}$$

(ii) If $Y=0$

$$\lim_{u\to\infty}\frac{\mathrm{Re}[\log\phi(u)]}{\log u}=-2C\tag{7.16}$$

(iii) If $Y<0$

$$\lim_{u\to\infty}\mathrm{Re}[\log\phi(u)]=-C\Gamma(-Y)(M^Y+G^Y)\tag{7.17}$$

Proof (see Appendix C)

Remark 6 It holds that

$$\lim_{Y\to 1}2C\Gamma(-Y)\cos\left(\frac{\pi}{2}Y\right)=-\pi C\tag{7.18}$$

7.3 Estimation

The CGMY processes have moments of any orders. So we can apply the classical method of moments to the estimation problems of the CGMY processes, but the equations obtained in the classical MM are not easy to solve. And so we try to to combine the characteristic function method with the classical MM. Using the above results, we can carry on the following estimation procedures.

Judgement of $Y>0$ or $Y\le 0$:

We can use the result of Proposition 1 for the judgement of the condition that $Y>0$ or $Y\le 0$. If the value

$$\frac{\log|\mathrm{Re}[\log\hat{\phi}(u)]|}{\log u}\tag{7.19}$$

is positive and separated from 0 for large u, we judge as $Y > 0$. If not, we judge as $Y \cdot 0$.

(i) Case of $Y > 0$.

Assume that we have judged that $Y > 0$. Then we can continue the following procedure.

1) Estimation of of the value of Y:

Using Proposition 1 again, we obtain an estimator of Y, $\hat{Y}$

$$\hat{Y} = \frac{\log j\mathrm{Re}[\log \hat{Â}(u_1)]j}{\log u_1}; \qquad (7.20)$$

where u_1 is a number large enough.

2) Estimation of b_1:

It is known that $b_1 = m_1$ (=mean). Therefore we obtain the estimator

$$\hat{b}_1 = \hat{h}_1 \quad (= \hat{m}_1) \qquad (7.21)$$

3) Estimation of C:

(1) The case $0 < Y < 2; Y \neq 1$:

By the result (i) of Proposition 3, we obtain the estimator $\hat{C}$ solving the following equation

$$\frac{\mathrm{Re}[\log \hat{Â}(u_3)]}{(u_3)^{\hat{Y}}} = 2C ¡ (¡ \hat{Y}) \cos \left(\frac{¼}{2}\hat{Y} \right) \qquad (7.22)$$

for some large number u_3.

(2) The case of $Y = 1$.

By the result (i) of Proposition 3, we obtain the estimator $\hat{C}$ solving the following equation

$$\frac{\mathrm{Re}[\log \hat{Â}(u_3)]}{(u_3)^{\hat{Y}}} = ¡ ¼ C: \qquad (7.23)$$

for some large number u_3.

4) Estimation of G and M:

Let $\hat{Y}$, $\hat{b}_1$ and $\hat{C}$ be the estimators of Y, b_1 and C respectively.

(1) The case $0 < Y < 2; Y \neq 1$: The moment equations (2.8) for $k = 1; 2$ are in this case

$$\hat{C}\Gamma(-\hat{Y})\hat{Y}(\hat{Y}-1)\left(G^{\hat{Y}-2} + M^{\hat{Y}-2}\right) = \hat{h}_2 \qquad (7.24)$$

$$\hat{C}\Gamma(-\hat{Y})\hat{Y}(\hat{Y}-1)(\hat{Y}-2)\left(G^{\hat{Y}-3} - M^{\hat{Y}-3}\right) = \hat{h}_3: \qquad (7.25)$$

We can solve this equation for G and M, and we obtain the estimators $\hat{G}$ and $\hat{M}$.

(2) The case of $Y = 1$: The moment equations (2.8) are in this case

$$\hat{C}\left(\frac{1}{G} + \frac{1}{M}\right) = \hat{h}_2 \qquad (7.26)$$

$$-\hat{C}\left(\frac{1}{G^2} - \frac{1}{M^2}\right) = \hat{h}_3 \qquad (7.27)$$

We can solve this equation for G and M, and we obtain the estimators $\hat{G}$ and $\hat{M}$.

(ii) Case of $Y \leq 0$.

Judgement of $Y = 0$ or $Y < 0$.

When we have judged as $Y \leq 0$, then we have to check whether $Y = 0$ or $Y < 0$. This is done by the use of the results of (ii) and (iii) of proposition 2.

If the limit

$$\lim_{u \to \infty} \mathrm{Re}[\log \phi(u)] \qquad (7.28)$$

diverged to $-\infty$, then we judge as $Y = 0$. If it converges, then we judge as $Y < 0$.

(iii) Case of $Y = 0$.

1) Estimation of C:

Using the following formula

$$\lim_{u \to \infty} \frac{\mathrm{Re}[\log \phi(u)]}{\log u} = -2C \qquad (7.29)$$

we obtain an estimator $\hat{C}$ of C

$$\hat{C} = -\frac{1}{2}\frac{\mathrm{Re}[\log \hat{\phi}(u_1)]}{u_1}; \qquad (7.30)$$

for some large number u_1.

2) Estimation of b_1:

The estimator $\hat{b}_1$ of b_1 is given by

$$\hat{b}_1 = \hat{h}_1 \tag{7.31}$$

3) Estimation of c_1 and c_2:

Let $\hat{C}$ and $\hat{b}_0$ be the estimators of C and b_0 respectively.. The moment equations (2.8) are in this case

$$\hat{C}\left(\frac{1}{G^2} + \frac{1}{M^2}\right) = \hat{h}_2 \tag{7.32}$$

$$-2\hat{C}\left(\frac{1}{G^3} - \frac{1}{M^3}\right) = \hat{h}_3: \tag{7.33}$$

We can solve this equation for c_1 and c_2.

(iv) Case of $Y < 0$.

1) Estimation of b_1:

$$\hat{b}_1 = \hat{h}_1 \quad (= \hat{m}_1) \tag{7.34}$$

2) Estimation of b_1:

The estimator $\hat{b}_1$ of b_1 is given by

$$\hat{b}_1 = \hat{h}_1 \tag{7.35}$$

3) Estimation of Y, C, G and M:

Let $\hat{b}_1$ be the estimator of b_1 . By (iii) of Proposition 3, we obtain an equation

$$\mathrm{Re}[\log \hat{A}(u_3)] = -C-(-Y)(G^Y + M^Y) \tag{7.36}$$

for some large u_3.

The moment equations (2.8) are in this case

$$C-(-Y)Y(Y-1)\left(G^{Y-2} + M^{Y-2}\right) = \hat{h}_2 \tag{7.37}$$

$$C-(-Y)Y(Y-1)(Y-2)\left(G^{Y-3} - M^{Y-3}\right) = \hat{h}_3 \tag{7.38}$$

$$C-(-Y)Y(Y-1)(Y-2)(Y-3)\left(G^{Y-4} + M^{Y-4}\right) = \hat{h}_4 \tag{7.39}$$

Solving the above 4 equations for $Y; C; G; M$, we obtain the estimators $\hat{Y}; \hat{C}; \hat{G}; \hat{M}$.

Appendix

A Characteristic Function of CGMY

(1) The case $Y = 0$.

This case is Variance Gamma case.

$$
\begin{aligned}
&\int_0^\infty (e^{-ux} - 1)x^{-1}e^{-Mx}dx \\
&= -\int_0^\infty \left(\int_0^u e^{-yx}dy\right) e^{-Mx}dx \\
&= -\int_0^u \left(\int_0^\infty e^{-(M+y)x}dx\right) dy \\
&= -\int_0^u \frac{1}{M+y}dy \\
&= -\log(M+U) - \log M \\
&= -\log(1+\frac{u}{M}) \qquad \text{(A.1)}
\end{aligned}
$$

$$\int_0^\infty (e^{iux} - 1)x^{-1}e^{-Mx}dx = -\log(1 - \frac{iu}{M}) \qquad \text{(A.2)}$$

The characteristic function $\phi_{VG}(u)$ is

$$
\begin{aligned}
\phi_{VG}(u) &= \exp\left\{ ib_0 u + \int_{|x|>0} (e^{iux} - 1)\nu(dx) \right\} \\
&= \exp\left\{ ib_0 u - C\left(\log(1+\frac{iu}{G}) + \log(1-\frac{iu}{M})\right)\right\} \\
&= e^{ib_0 u}\left(\frac{1}{\left(1+\frac{iu}{G}\right)\left(1-\frac{iu}{M}\right)}\right)^C \qquad \text{(A.3)}
\end{aligned}
$$

(2) The case $Y < 1$ and $Y \neq 0$.

We can adopt the generating triplet $(0, \nu, b_0)_0$, and the characteristic function is

$$\phi_{CGMY}(u) = \exp\left(ib_0 u + \int_{|x|>0} (e^{iux} - 1)\nu(dx) \right) \qquad \text{(A.4)}$$

For $u>0$, we can calculate

$$\begin{aligned}
&\int_0^\infty (e^{-ux}-1)x^{-1-Y}e^{-Mx}dx \\
&= -\int_0^\infty \left(\int_0^u e^{-yx}dy\right) x^{-Y}e^{-Mx}dx \\
&= -\int_0^u \left(\int_0^\infty x^{-Y}e^{-(M+y)x}dx\right) dy \\
&= -\int_0^u (M+y)^{Y-1}\left(\int_0^\infty v^{-Y}e^{-v}dv\right) dy \\
&= -\Gamma(1-Y)\frac{1}{Y}\left((M+u)^Y - M^Y\right) \\
&= \Gamma(-Y)\left((M+u)^Y - M^Y\right)
\end{aligned} \tag{A.5}$$

$$\int_0^\infty (e^{iux}-1)x^{-1-Y}e^{-Mx}dx = \Gamma(-Y)\left((M-iu)^Y - M^Y\right) \tag{A.6}$$

Therefore we obtain

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_0u + \int_{|x|>0}(e^{iux}-1)\nu(dx)\right\} \\
&= \exp\left\{ ib_0u + C\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)\right\}
\end{aligned} \tag{A.7}$$

If we adopt another expression of the generating triplet, $(0, \nu(dx), b_1)_1$, then we obtain

$$\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_1u + \int_{|x|>0}(e^{iux}-1-iux)\nu(dx)\right\} \\
&= \exp\left\{ i\left(b_1 - \int_{|x|>0} x\nu(dx)\right)u + \int_{|x|>0}(e^{iux}-1)\nu(dx)\right\} \\
&= \exp\Big\{ i\left(b_1 - C\Gamma(1-Y)(M^{Y-1}-G^{Y-1})\right)u \\
&\quad + C\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)\Big\} \\
&= \exp\Big\{ i\left(b_1 + C\Gamma(-Y)Y(M^{Y-1}-G^{Y-1})\right)u \\
&\quad + C\Gamma(-Y)\left((M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right)\Big\}
\end{aligned} \tag{A.8}$$

(3) The case $Y=1$.

$$\int_0^\infty (e^{iux}-1-iux)x^{-2}e^{-Mx}dx$$

$$
\begin{aligned}
&= \int_0^\infty (e^{iux} - 1 - iux)\left(\frac{d\left\{-x^{-1}e^{-Mx}\right\}}{dx} - Mx^{-1}e^{-Mx}\right)dx \\
&= -\left(\left[(e^{iux} - 1 - iux)(x^{-1}e^{-Mx})\right]_0^\infty - iu\int_0^\infty (e^{iux} - 1)(x^{-1}e^{-Mx})dx\right) \\
&\quad - M\int_0^\infty (e^{iux} - 1 - iux)(-x^{-1}e^{-Mx})dx \\
&= iu\int_0^\infty (e^{iux} - 1)(x^{-1}e^{-Mx})dx \\
&\quad - M\left(\int_0^\infty (e^{iux} - 1)(x^{-1}e^{-Mx})dx - iu\int_0^\infty e^{-Mx}dx\right) \\
&= (iu - M)\int_0^\infty (e^{iux} - 1)(x^{-1}e^{-Mx})dx + iuM\int_0^\infty e^{-Mx}dx \\
&= -(iu - M)\log\left(1 - \frac{iu}{M}\right) + iu; \qquad \text{(A.9)}
\end{aligned}
$$

where we use the following formula

$$
\int_0^\infty (e^{iux} - 1)x^{-1}e^{-Mx}dx = -\log\left(1 - \frac{iu}{M}\right) \qquad \text{(A.10)}
$$

Therefore we obtain

$$
\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ib_1u + \int_{|x|>0}(e^{iux} - 1 - iux)\nu(dx)\right\} \\
&= \exp\left\{ib_1u + C\left(\left((M - iu)\log\left(1 - \frac{iu}{M}\right) + iu\right) + \left((G + iu)\log\left(1 + \frac{iu}{G}\right) - iu\right)\right)\right\} \\
&= \exp\left\{ib_1u + C\left((M - iu)\log\left(1 - \frac{iu}{M}\right) + (G + iu)\log\left(1 + \frac{iu}{G}\right)\right)\right\} \qquad \text{(A.11)}
\end{aligned}
$$

(4) The case $1 < Y < 2$.

$$
\begin{aligned}
&\int_0^\infty (e^{iux} - 1 - iux)x^{-1-Y}e^{-Mx}dx \\
&= \int_0^\infty (e^{iux} - 1 - iux)\frac{1}{Y}\left(-\frac{d\left\{x^{-Y}e^{-Mx}\right\}}{dx} - Mx^{-Y}e^{-Mx}\right)dx \\
&= -\frac{1}{Y}\left(\left[(e^{iux} - 1 - iux)(x^{-Y}e^{-Mx})\right]_0^\infty - iu\int_0^\infty (e^{iux} - 1)(x^{-Y}e^{-Mx})dx\right) \\
&\quad - \frac{M}{Y}\int_0^\infty (e^{iux} - 1 - iux)(x^{-Y}e^{-Mx})dx \\
&= \frac{iu}{Y}\int_0^\infty (e^{iux} - 1)(x^{-Y}e^{-Mx})dx \\
&\quad - \frac{M}{Y}\left(\int_0^\infty (e^{iux} - 1)(x^{-Y}e^{-Mx})dx - iu\int_0^\infty x^{1-Y}e^{-Mx}dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{iu - M}{Y}\int_0^\infty (e^{iux} - 1)(x^{-Y}e^{-Mx})dx + \frac{iuM}{Y}\int_0^\infty x^{1-Y}e^{-Mx}dx \\
&= \frac{iu - M}{Y}\Gamma(1-Y)\left((M - iu)^{Y-1} - M^{Y-1}\right) + \frac{iuM}{Y}\Gamma(2-Y)M^{Y-2} \\
&= \Gamma(-Y)(M - iu)\left((M - iu)^{Y-1} - M^{Y-1}\right) - iu\Gamma(-Y)(1-Y)M^{Y-1} \\
&= \Gamma(-Y)\left((M - iu)^{Y} - M^{Y}\right) + iu\Gamma(-Y)YM^{Y-1} \qquad \text{(A.12)}
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
\phi_{CGMY}(u) &= \exp\left\{ ib_1u + \int_{|x|>0}(e^{iux} - 1 - iux)\nu(dx)\right\} \\
&= \exp\Big\{ ib_1u + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right) \\
&\quad + iuC\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\Big\} \\
&= \exp\Big\{ i\left(b_1 + C\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\right)u \\
&\quad + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\Big\} \qquad \text{(A.13)}
\end{aligned}
$$

B Proof of Proposition 2

(Proof)

(1) The case of $0 < Y < 2$, $Y \neq 1$:

$$
\begin{aligned}
\phi_{CGMY}(u) &= \exp\Big\{ i\left(b_1 + C\Gamma(-Y)Y\left(M^{Y-1} - G^{Y-1}\right)\right)u \\
&\quad + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right)\Big\} \qquad \text{(B.1)}
\end{aligned}
$$

Set

$$M - iu = \left(\sqrt{M^2 + u^2}\right) e^{i\theta_M(u)} \qquad \text{(B.2)}$$

$$G + iu = \left(\sqrt{G^2 + u^2}\right) e^{i\theta_G(u)} \qquad \text{(B.3)}$$

Remark that $\theta_M(u)$ is a decreasing function and $\theta_G(u)$ is an increasing function, and that

$$\lim_{u\to\infty} \theta_M(u) = -\frac{\pi}{2} \qquad \text{(B.4)}$$

$$\lim_{u\to\infty} \theta_G(u) = \frac{\pi}{2}. \qquad \text{(B.5)}$$

$$\mathrm{Re}[\log \phi(u)] = C\Gamma(-Y)\left(\left(\sqrt{M^2+u^2}\right)^Y \cos(Y\theta_M(u)) - M^Y + \left(\sqrt{G^2+u^2}\right)^Y \cos(Y\theta_G(u)) - G^Y\right). \quad \text{(B.6)}$$

$$\lim_{u\to\infty} \cos(Y\theta_M(u)) = \cos\left(-\frac{\pi}{2}Y\right) = \cos\left(\frac{\pi}{2}Y\right) \quad \text{(B.7)}$$

$$\lim_{u\to\infty} \cos(Y\theta_G(u)) = \cos\left(\frac{\pi}{2}Y\right) \quad \text{(B.8)}$$

If $0 < Y < 1$, then $\cos\left(\frac{\pi}{2}Y\right)$ is positive. If $1 < Y < 2$, then $\cos\left(\frac{\pi}{2}Y\right)$ is negative. From these facts it follows that $|\mathrm{Re}[\log \phi(u)]|$ is of order $|u|^Y$ when $u \to \infty$. So we have obtained the result (7.14).

(2) The case $Y = 1$:

$$\phi(u) = \exp\left\{ib_1 u + C\left((M - iu)\log(1 - \frac{iu}{M}) + (G + iu)\log(1 + \frac{iu}{G})\right)\right\} \quad \text{(B.9)}$$

Using the same notations as above, we obtain

$$\begin{aligned}\log \phi(u) &= ib_1 u + C\left(\sqrt{M^2+u^2}\,e^{i\theta_M(u)}\left(\log\sqrt{1+\frac{u^2}{M^2}} + i\theta_M(u)\right)\right.\\ &\quad \left. + \sqrt{G^2+u^2}\,e^{i\theta_G(u)}\left(\log\sqrt{1+\frac{u^2}{G^2}} + i\theta_G(u)\right)\right) \quad \text{(B.10)}\end{aligned}$$

$$\begin{aligned}&\mathrm{Re}[\log \phi(u)]\\ &= C\left(\sqrt{M^2+u^2}\left(\log\sqrt{1+\frac{u^2}{M^2}}\cos\theta_M(u) - \theta_M(u)\sin\theta_M(u)\right)\right.\\ &\quad \left. + \sqrt{G^2+u^2}\left(\log\sqrt{1+\frac{u^2}{G^2}}\cos\theta_G(u) - \theta_G(u)\sin\theta_G(u)\right)\right)\\ &= C\left(M\log\sqrt{1+\frac{u^2}{M^2}} + \theta_M(u)u + G\log\sqrt{1+\frac{u^2}{G^2}} - \theta_G(u)u\right) \quad \text{(B.11)}\end{aligned}$$

where we use

$$\cos\theta_M(u) = \frac{M}{\sqrt{M^2+u^2}}; \qquad \sin\theta_M(u) = \frac{-u}{\sqrt{M^2+u^2}} \quad \text{(B.12)}$$

$$\cos\theta_G(u) = \frac{G}{\sqrt{G^2+u^2}}; \qquad \sin\theta_G(u) = \frac{u}{\sqrt{G^2+u^2}} \quad \text{(B.13)}$$

The formula (7.14) follows from (B.11).

(3) $Y = 0$

In this case

$$\phi_{VG}(u) = \exp\left\{ i b_0 u - C\left(\log(1+\frac{iu}{G}) + \log(1-\frac{iu}{M}) \right) \right\} \tag{B.14}$$

Therefore

$$\begin{aligned} & \mathrm{Re}[\log \phi(u)] \\ = & -C\left(\log \sqrt{1+\frac{u^2}{M^2}} + \log \sqrt{1+\frac{u^2}{G^2}} \right) \end{aligned} \tag{B.15}$$

The formula (7.14) follows from (B.15).

(4) $Y < 0$

In this case

$$\phi(u) = \exp\left\{ i b_0 u + C\Gamma(-Y)\left((M - iu)^Y - M^Y + (G+iu)^Y - G^Y \right) \right\} \tag{B.16}$$

and

$$\begin{aligned} \mathrm{Re}[\log \phi(u)] = \; & C\Gamma(-Y)\left(\left(\sqrt{M^2+u^2}\right)^Y \cos(Y\theta_M(u)) - M^Y \right. \\ & \left. + \left(\sqrt{G^2+u^2}\right)^Y \cos(Y\theta_G(u)) - G^Y \right). \end{aligned} \tag{B.17}$$

Since $Y < 0$, it is easy to see the formula (7.14).

(Q.E.D)

C Proof of Proposition 3

(Proof)

(i) (1) The case of $0 < Y < 2, Y \neq 1$

$$\begin{aligned} \mathrm{Re}[\log \phi(u)] = \; & C\Gamma(-Y)\left(\left(\sqrt{M^2+u^2}\right)^Y \cos(Y\theta_M(u)) - M^Y \right. \\ & \left. +(\left(\sqrt{G^2+u^2}\right)^Y \cos(Y\theta_G(u)) - G^Y \right). \end{aligned} \tag{C.1}$$

$$\lim_{u \to \infty} \cos(Y\,\theta_M(u)) = \cos\left(-\frac{\pi}{2}Y\right) = \cos\left(\frac{\pi}{2}Y\right) \tag{C.2}$$

$$\lim_{u \to \infty} \cos(Y\,\theta_G(u)) = \cos\left(\frac{\pi}{2}Y\right) \tag{C.3}$$

Using these formulae, we obtain

$$\lim_{u \to \infty} \frac{\mathrm{Re}[\log \phi(u)]}{u^Y} = 2C\,\Gamma(-Y)\cos\left(\frac{\pi}{2}Y\right) : \tag{C.4}$$

(2) The case $Y = 1$:

$$\mathrm{Re}[\log \phi(u)] = C\left(M \log \sqrt{1 + \frac{u^2}{M^2}} + \theta_M(u)u + G \log \sqrt{1 + \frac{u^2}{G^2}} - \theta_G(u)u\right) \tag{C.5}$$

and

$$\lim_{u \to \infty} \frac{\mathrm{Re}[\log \phi(u)]}{u} = -\pi C: \tag{C.6}$$

(ii) The case $Y = 0$:

$$\mathrm{Re}[\log \phi(u)] = -C\left(\log \sqrt{1 + \frac{u^2}{M^2}} + \log \sqrt{1 + \frac{u^2}{G^2}}\right) \tag{C.7}$$

and so

$$\lim_{u \to \infty} \frac{\mathrm{Re}[\log \phi(u)]}{\log u} = -2C \tag{C.8}$$

(iii) The case $Y < 0$:

$$\begin{aligned} \mathrm{Re}[\log \phi(u)] &= C\,\Gamma(-Y)\left(\left(\sqrt{M^2 + u^2}\right)^Y \cos(Y\,\theta_M(u)) - M^Y \right. \\ &\quad \left. + \left(\sqrt{G^2 + u^2}\right)^Y \cos(Y\,\theta_G(u)) - G^Y\right) : \end{aligned} \tag{C.9}$$

Since $Y < 0$

$$\lim_{u \to \infty} \mathrm{Re}[\log \phi(u)] = -C\,\Gamma(-Y)(M^Y + G^Y) \tag{C.10}$$

(Q.E.D.)

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