A Note on Esscher Transformed Martingale Measures for Geometric Lévy Processes

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Abstract

The Esscher transform is one of the very useful methods to obtain the reasonable equivalent martingale measures, and it is defined with relation to the corresponding risk process.

In this article we consider two kinds of risk processes (compound return process and simple return process). Then we obtain two kinds of Esscher transformed martingale measures. The first one is the one which was introduced by Gerber and Shiu, and the second one is identified with the MEMM (minimal entropy martingale measure).

We set up the economical characterization of these two kinds of Esscher transforms, and then we study the properties of the above two kinds of Esscher transformed martingale measures, comparing each others.

Key words: incomplete market, geometric Lévy process, equivalent martingale measure, Esscher transform, minimal entropy martingale measure.
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1 Introduction

The equivalent martingale measure method is one of the most powerful methods in the option pricing theory. If the market is complete, then the equivalent martingale measure is unique. On the other hand, in the incomplete market model there are many equivalent martingale measures. So we have to select one equivalent martingale measure (EMM) as the suitable martingale measure in order to apply the martingale measure method.

In the case of geometric Lévy process models, four kinds of measures have been proposed. The first one is the minimal martingale measure (MMM), the second one is the Esscher martingale measure (ESSMM), the third one is the minimal entropy martingale measure (MEMM) , and the fourth one is the utility martingale measure (UMM).

The Esscher transform is thought to be a very useful technique to obtain the reasonable equivalent martingale measure, and it is defined in the relation to the corresponding risk process. (See [6], [10], [1], [14], etc.) In this article we consider two kinds of risk processes (compound return process and simple return process).

According to these risk processes, we obtain two kinds of Esscher transformed martingale measures. The first one is the compound return Esscher transformed martingale measure, which was first introduced by Gerber and Shiu ([10]), and this measure is called the "Esscher martingale measure (ESSMM)". The second one is the simple return Esscher transformed martingale measure. It is known that this martingale measure is identified with the "minimal entropy martingale measure (MEMM)." (See [9] or [14]). The MEMM has been discussed in [15], [7], [9], and etc.

In §2 we explain the geometric Lévy process and in §3 we give the definition of two kinds of Esscher transforms. In §4 we study the economical implications of those transforms. In §5 we summarize the existence conditions of those martingale measures, and finally in §6 we review the properties of ESSMM and MEMM.

As the results of our consideration, we can say that the MEMM is very effective to the option pricing theory from the theoretical point of view.

2 Geometric Lévy processes

The price process S_t of a stock is assumed to be defined as what follows. We suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t, 0 \leq t \leq T\}$ are given, and that the price process $S_t = S_0 e^{Z_t}$ of a stock is defined on this probability space, where Z_t is a Lévy process. We call such a process S_t the geometric Lévy precess (GLP).

Throughout this paper we assume that $\mathcal{F}_t = \sigma(S_s, 0 \leq s \leq t) = \sigma(Z_s, 0 \leq s \leq t)$ and $\mathcal{F} = \mathcal{F}_T$. A probability measure Q on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure of S_t if $Q \sim P$ and $e^{-rt}S_t$ is $(\mathcal{F}_t, Q)$-martingale, where r is the interest rate.

The price process S_t has the other expression

$$S_t = S_0 \mathcal{E}(\tilde{Z})_t \tag{2.1}$$

where $\mathcal{E}(\tilde{Z})_t$ is the Doléans-Dade exponential of $\tilde{Z}_t$, and $\tilde{Z}_t$ is a Lévy process corresponding to the original Lévy process Z_t. Let the generating triplet of Z_t is $(\sigma^2, \nu(dx), b)$, then the generating triplet of $\tilde{Z}_t$, say $(\tilde{\sigma}^2, \tilde{\nu}(dx), \tilde{b})$, is

$$\begin{aligned} \tilde{\sigma}^2 &= \sigma^2 && (2.2)\\ \tilde{\nu}(dx) &= (\nu \circ J^{-1})(dx), \quad J(x) = e^x - 1, && (2.3)\\ \tilde{b} &= b + \frac{1}{2}\sigma^2 + \int_{\{|x| \leq 1\}} (e^x - 1 - x)\nu(dx) \\ &\quad + \int_{\{x < -1\}} (e^x - 1)\nu(dx) - \int_{\{\log 2 < x \leq 1\}} (e^x - 1)\nu(dx). && (2.4) \end{aligned}$$

Remark 1 (i) *It holds that supp* $\tilde{\nu} \subset (-1, \infty)$.
(ii) *If* $\nu(dx)$ *has the density* $n(x)$*, then* $\tilde{\nu}(dx)$ *has the density* $\tilde{n}(x)$ *and* $\tilde{n}(x)$ *is given by*

$$\tilde{n}(x) = \frac{1}{1+x} n(\log(1+x)). \tag{2.5}$$

(iii) S_t *satisfies the following stochastic differential equation*

$$dS_t = S_{t-} d\tilde{Z}_t. \tag{2.6}$$

(iv) *The relations between* Z_t *and* $\tilde{Z}_t$ *are more precisely discussed in* [14].

3 Esscher transforms

The Esscher transform is very popular in the field of actuary, and is thought to be very important method in the actuary theory. (See [10]). Esscher has introduced the risk function and the transformed risk function for the calculation of collective risk in [6]. His idea has been developed by many researchers ([10], [1], [14]), and applied to the option pricing theory.

We give the definitions of Esscher transform and Esscher transformed martingale measure.

Definition 1 *Let R be a risk variable and h be a constant. Then the probability measure $P^{(ESS)}_{R,h}$ defined by*

$$\frac{dP^{(ESS)}_{R,h}}{dP}|_{\mathcal{F}} = \frac{e^{hR}}{E[e^{hR}]} \tag{3.1}$$

is called the Esscher transformed measure of P by the random variable R and h, and this measure transformation is called the Esscher transform by the random variable R and h.

Definition 2 *Let $R_t, 0 \leq t \leq T$, be a risk process. Then the Esscher transformed measure of P by the process R_t and a constant h is the probability measure $P^{(ESS)}_{R_{[0,T]},h}$, which is defined by*

$$\frac{dP^{(ESS)}_{R_{[0,T]},h}}{dP}|_{\mathcal{F}} = \frac{e^{hR_T}}{E[e^{hR_T}]} \tag{3.2}$$

(*Remark that $P^{(ESS)}_{R_{[0,T]},h} = P^{(ESS)}_{R_T,h}$.*)

and this measure transformation is called the Esscher transform by the process R_t and a constant h.

Definition 3 *In the above definition, if the constant h is chosen so that the $P^{(ESS)}_{R_{[0,T]},h}$ is a martingale measure of S_t, then $P^{(ESS)}_{R_{[0,T]},h}$ is called the Esscher transformed martingale measure of S_t by the process R_t, and it is denoted by $P^{(ESS)}_{R_{[0,T]}}$ or $P^{(ESS)}_{R_T}$.*

4 Esscher transformed martingale measures for geometric Lévy processes

4.1 Simple return process and compound return process

When we give a certain risk process R_t, we obtain a corresponding Esscher transformed martingale measure if it exists. As we have seen in the previous section, the GLP has two kinds of representation such that

$$S_t = S_0 e^{Z_t} = S_0 \mathcal{E}(\tilde{Z})_t.$$

The processes Z_t and $\tilde{Z}_t$ are candidates for the risk process.

We shall see the economical meaning of them. For this purpose, we will review the discrete time approximation of geometric Lévy processes.

Set

$$S_k^{(n)} = S_{k/2^n}, \quad k = 1, 2, \ldots. \tag{4.1}$$

According to the above two kinds of expression of S_t, we obtain two kinds of approximation formula.

First one is

$$S_k^{(n)} = S_0 e^{Z_k^{(n)}}, \quad k = 1, 2, \ldots, \tag{4.2}$$

where $Z_k^{(n)} = Z_{k/2^n}$.

Second approximation is

$$S_k^{(n)} = S_0 \mathcal{E}(Y^{(n)})_k, \quad k = 1, 2, \ldots, \tag{4.3}$$

where $\mathcal{E}(Y^{(n)})_k$ is the discrete time Doléans-Dade exponential of $Y_k^{(n)}$,

$$\mathcal{E}(Y^{(n)})_k = \prod_{j=1}^{k} \left(1 + (Y_j^{(n)} - Y_{j-1}^{(n)})\right) \tag{4.4}$$

and $Y_k^{(n)}$ is defined from the following relations

$$e^{Z_k^{(n)}} = \mathcal{E}(Y^{(n)})_k = \prod_{j=1}^{k} \left(1 + (Y_j^{(n)} - Y_{j-1}^{(n)})\right), \quad k = 1, 2, \ldots. \tag{4.5}$$

So we obtain

$$e^{\triangle Z_k^{(n)}} = e^{Z_k^{(n)} - Z_{k-1}^{(n)}} = \left(1 + (Y_k^{(n)} - Y_{k-1}^{(n)})\right) = 1 + \triangle Y_k^{(n)}. \tag{4.6}$$

From this we obtain

$$\frac{\triangle S_k^{(n)}}{S_{k-1}^{(n)}} = \frac{S_k^{(n)} - S_{k-1}^{(n)}}{S_{k-1}^{(n)}} = \frac{S_k^{(n)}}{S_{k-1}^{(n)}} - 1 = e^{\triangle Z_k^{(n)}} - 1 = \triangle Y_k^{(n)} \tag{4.7}$$

and we know that $\triangle Y_k^{(n)}$ is the simple return process of $S_k^{(n)}$.

On the other hand, we obtain the following formula for $\triangle Z_k^{(n)}$

$$\triangle Z_k^{(n)} = \log\left(1 + \triangle Y_k^{(n)}\right) = \log\left(1 + \frac{\triangle S_k^{(n)}}{S_{k-1}^{(n)}}\right), \tag{4.8}$$

and we know that $\triangle Z_k^{(n)}$ is the compound return process of $S_k^{(n)}$.

Remark 2 *The terms* 'simple return' *and* 'compound return' *were introduced in* [1, p.294].

For $t \in (\frac{k-1}{2^n}, \frac{k}{2^n})$ we define

$$Z_t^{(n)} = Z_k^{(n)}, \quad Y_t^{(n)} = Y_k^{(n)}. \tag{4.9}$$

It is easy to see that the process $Z_t^{(n)}$ converges to the process Z_t when n goes to ∞.

On the other hand we can see that the process $Y_t^{(n)}$ converges to the process $\tilde{Z}_t$. As we have seen, S_t satisfies the following stochastic differential equation

$$dS_t = S_{t-} d\tilde{Z}_t. \tag{4.10}$$

From this it follows that

$$d\tilde{Z}_t = \frac{dS_t}{S_{t-}}. \tag{4.11}$$

Comparing the formulae (4.7) and (4.11), we know that the process $Y_t^{(n)}$ is the approximation process in the procedure of solving the equation (4.11) for $\tilde{Z}_t$. This fact means that the process $Y_t^{(n)}$ converges to the process $\tilde{Z}_t$.

Based on the above observation, it is natural for us to give the following definition.

Definition 4 *The process $\tilde{Z}_t$ is called the simple return process of S_t, and the process Z_t is called the compound return process of S_t.*

4.2 Two kinds of Esscher transformed martingale measures

Suppose that Z_t is adopted as the risk process. In this case if the corresponding Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$ is well defined, then it should be called the *'compound return Esscher transformed martingale measure'*. This is the Gerber-Shiu's Esscher martingale measure introduced in [10], and the term *'Esscher martingale measure'* is usually suggesting this compound return Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$.

Next we consider the case where $\tilde{Z}_t$ is adopted as the risk process. If the corresponding Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ exists, then it should be called the *'simple return Esscher transformed martingale measure'*.

In [14] the following results have been obtained.

Proposition 1 ([14], Theorem 4.2) *The compound return Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$ is unique if it exists.*

Proposition 2 ([14], Theorem 4.5) *The simple return Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ is unique if it exists.*

From the proof of [9, Theorem 3.1] it follows that

Proposition 3 *The simple return Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ of S_t is the minimal entropy martingale measure* (MEMM) *of S_t.*

Based on the above results, we give the following definition.

Definition 5 (i) *The compound return Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$ is called the 'Esscher martingale measure* (ESSMM)*' and denoted by $P^{(ESSMM)}$.*
(ii) *The simple return Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ is called the 'minimal entropy martingale measure* (MEMM)*' and denoted by P^* (or $P^{(MEMM)}$).*

Remark 3 *For the jump-diffusion models, the Brownian motion can be adopted as the risk process. In that case the corresponding Esscher transformed martingale measure is the mean correcting martingale measure.* (See [21] or [3]).

5 Existence theorems of ESSMM and MEMM for geometric Lévy processes

The uniqueness theorems are stated in the previous section. We next study the Existence problem of Esscher transformed martingale measures.

5.1 Existence theorem of ESSMM

We suppose that the expectations which appear in what follows exist. Then the martingale condition for an Esscher transformed probability measure $Q = P^{(ESS)}_{Z_{[0,T]},h}$ is

$$E_Q[e^{-r}S_1] = e^{-r}S_0E_Q[e^{Z_1}] = e^{-r}S_0\frac{E_P[e^{(h+1)Z_1}]}{E_P[e^{hZ_1}]} = S_0. \tag{5.1}$$

This condition is equal to the following condition

$$E_P[e^{(h+1)Z_1}] = e^r E_P[e^{hZ_1}], \tag{5.2}$$

and this is also equivalent to the following expression,

$$\phi(-i(h+1)) = e^r\phi(-ih), \quad \phi(u) = E_P[e^{iuZ_1}], \tag{5.3}$$

where $\phi(u) =$ is the characteristic function of Z_1.

To formulate the existence theorem, we set

$$\begin{aligned} f^{(ESSMM)}(h) &= b + (\frac{1}{2}+h)\sigma^2 + \int_{\{|x|\le 1\}}\left((e^x-1)e^{hx} - x\right)\nu(dx) \\ &\quad + \int_{\{|x|>1\}}(e^x-1)e^{hx}\,\nu(dx), \end{aligned} \tag{5.4}$$

Then we obtain

Theorem 1 (Existence condition for ESSMM) *If the equation*

$$f^{(ESSMM)}(h) = r, \tag{5.5}$$

has a solution h^*, then the ESSMM of S_t, $P^{(ESSMM)}$, exists and

$$P^{(ESSMM)} = P^{(ESS)}_{Z_{[0,T]},h^*} = P^{(ESS)}_{Z_T,h^*} \tag{5.6}$$

The process Z_t is also a Lévy process under $P^{(ESSMM)}$ and the generating triplet of Z_t under $P^{(ESSMM)}$, $(\sigma^{(ESSMM)^2}, \nu^{(ESSMM)}(dx), b^{(ESSMM)})$, is

$$\sigma^{(ESSMM)^2} = \sigma^2, \tag{5.7}$$

$$\nu^{(ESSMM)}(dx) = e^{h^* x}\nu(dx), \tag{5.8}$$

$$b^{(ESSMM)} = b + h^*\sigma^2 + \int_{\{|x|\le 1\}} x(e^{h^* x} - 1)\nu(dx). \tag{5.9}$$

(Proof) The equation (5.5) is equivalent to the condition (5.3). Therefore $P^{(ESS)}_{Z_{[0,T]},h^*}$ is a martingale measure of S_t.

The characteristic function of Z_t under $P^{(ESSMM)} = P^{(ESS)}_{Z_{[0,T]},h^*}$, $\phi_t^{(ESSMM)}(u)$, is by definition

$$\phi_t^{(ESSMM)}(u) = E_{P^{(ESSMM)}}[e^{iuZ_t}] = \frac{E_P[e^{iuZ_t}e^{h^* Z_T}]}{E_P[e^{h^* Z_T}]}. \tag{5.10}$$

And this is equal to

$$\frac{E_P[e^{(iu+h^*)Z_t}}{E_P[e^{h^* Z_t}]} = \frac{\phi_t(u - ih^*)}{\phi_t(-ih^*)}. \tag{5.11}$$

By simple calculation we obtain

$$\begin{aligned}\phi_t^{(ESSMM)}(u) = \ & \exp\Big\{t\Big(-\tfrac{1}{2}\sigma^2 + i(b + h^*\sigma^2 + \textstyle\int_{\{|x|\le 1\}} x(e^{h^* x} - 1)\nu(dx))u \\ & + \textstyle\int_{\{|x|\le 1\}}(e^{iux} - 1 - iux)e^{h^* x}\nu(dx) \\ & + \textstyle\int_{\{|x|>1\}}(e^{iux} - 1)e^{h^* x}\nu(dx)\Big)\Big\}.\end{aligned} \tag{5.12}$$

This formula proves the results of the theorem. (Q.E.D.)

5.2 Existence theorem of MEMM

As we have mentioned in the previous section, the MEMM, P^*, is the simple return Esscher transformed martingale measure. ($P^* = P^{(ESS)}_{\tilde{Z}_{[0,T]}}$). The existence theorem of the MEMM is obtained in [9].

Set

$$\begin{aligned} f^{(MEMM)}(\theta) = \ & b + (\frac{1}{2} + \theta)\sigma^2 + \int_{\{|x|\le 1\}} \left((e^x - 1)e^{\theta(e^x - 1)} - x\right)\nu(dx) \\ & + \int_{\{|x|>1\}} (e^x - 1)e^{\theta(e^x - 1)}\nu(dx) \end{aligned} \tag{5.13}$$

Then the following result is obtained ([9, Theorem 3.1]).

Theorem 2 (Existence condition for MEMM) *If the equation*

$$f^{(MEMM)}(\theta) = r \tag{5.14}$$

has a solution θ^**, then the MEMM of* S_t*,* P^**, exists and*

$$P^* = P^{(MEMM)} = P^{(ESS)}_{\tilde{Z}_{[0,T]},\theta^*} = P^{(ESS)}_{\tilde{Z}_T,\theta^*} \tag{5.15}$$

The process Z_t *is also a Lévy process under* P^* *and the generating triplet of* Z_t *under* P^**,* $(\sigma^{*2}, \nu^*(dx), b^*)$*, is*

$$\sigma^{*2} = \sigma^2, \tag{5.16}$$

$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx), \tag{5.17}$$

$$b^* = b + \theta^*\sigma^2 + \int_{\{|x|\leq 1\}} x(e^{\theta^*(e^x-1)} - 1)\nu(dx). \tag{5.18}$$

(Proof) The results of this theorem follows directly from the proof of [9, Theorem 3.1]. (Q.E.D.)

It is easy to see that the function $f^{(MEMM)}(\theta)$ is an increasing function of θ. Therefore, if $f^{(MEMM)}(\theta)$ is a continuous function and satisfies the following inequality

$$\lim_{\theta\to-\infty} f^{(MEMM)}(\theta) < r < \lim_{\theta\to\infty} f^{(MEMM)}(\theta), \tag{5.19}$$

then the equation (5.14) has a solution, and the MEMM exists.

6 Comparison of ESSMM and MEMM

The ESSMM and the MEMM are both obtained by Esscher transform, but they have different properties. We will survey the properties and the differences of them.

1) As we have seen in the previous section, for the existence of ESSMM, $P^{(ESSMM)}$, the following condition

$$\int_{\{|x|>1\}} |(e^x - 1)e^{h^*x}|\,\nu(dx) < \infty \tag{6.1}$$

is necessary. On the other hand, for the existence of MEMM, P^*, the corresponding condition is

$$\int_{\{|x|>1\}} |(e^x - 1)e^{\theta^*(e^x-1)}|\,\nu(dx) < \infty. \tag{6.2}$$

This condition is satisfied for wide class of Lévy measures, if $\theta^* < 0$. Namely, the former condition is strictly stronger than the latter condition. This means that the MEMM may be applied to the wider class of models than the ESSMM. The difference works in the stable process cases. In fact we can make sure that MEMM method can be applied to geometric stable models but ESSMM method can not be applied to this model.

2) The ESSMM is corresponding to power utility function or logarithm utility function. (See [10, pp.175-177] or [11, Corollary 6.3]). On the other hand the MEMM is corresponding to the exponential utility function. (See [7, §3] or [11, §6.1]).

We remark here that, in the case of ESSMM, the power parameter of the utility function depends on the parameter value h^* of the Esscher transform.

We also remark that, in the case of MEMM, the relation of the MEMM to the utility indifference price is known. (See [9, §4]. This result is generalized by C. Stricker [23].)

3) The relative entropy is very popular in the field of information theory, and it is called Kullback-Leibler Information Number(see [12, p.23]) or Kullback-Leibler distance (see [5, p.18]). Therefore we can state that the MEMM is the nearest equivalent martingale measure to the original probability P in the sense of Kullback-Leibler distance. Recently the idea of minimal distance martingale measure is studied. In [11] it is mentioned that the relative entropy is the typical example of the distance in their theory.

4) The large deviation theory is closely related to the minimum relative entropy analysis, and the Sanov's theorem or Sanov property is well-known (see, e.g. [5, p.291-304] or [12, p.110-111]). This theorem says that the MEMM is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures. In this sense the MEMM should be considered to be the exceptional measure in the class of all equivalent martingale measures.

7 Concluding Remarks

As we have seen the previous sections, the MEMM has many good properties and seems to be superior to ESSMM from the theoretical point of view. And we can say that the [GLP & MEMM] model, which has been introduced in [17], is a strong candidate for the incomplete market model.

What we should do next is to verify that the [GLP & MEMM] model is very useful in the actual world. To do this we have to carry out the calibration of this model. This task is now under progress. (See [18]).

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