The [GLP & MEMM] Pricing Model and its Calibration Problems

Yoshio Miyahara*
Graduate School of Economics
Nagoya City University

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Abstract

The [GLP & MEMM] pricing model (= [Geometric Lévy Process & Minimal Entropy Martingale Measure] pricing model) has been introduced as a pricing model for the incomplete financial market. This model has many good properties and is applicable to very wide classes of underlying asset price processes including the geometric stable processes. We explain those good properties and see several examples of this model. After that we investigate the calibration problems of [GLP & MEMM] model.

Key words: Geometric Lévy Process, Relative entropy, Minimal entropy martingale measure, Stable process, Calibration.
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*E-mail:y-miya@econ.nagoya-cu.ac.jp

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1 Introduction

Geometric Lévy process pricing models are discussed in relation to the incomplete markets.

The [Geometric Lévy Process & MEMM] pricing model was first introduced in [30]. This model is one of the incomplete markets, and is based on the geometric Lévy process and the minimal entropy martingale measure (= MEMM).

In Section 2, 3 and 4 we explain this model and see the properties of this model. In Section 5 we investigate the relations between the physical world and the MEMM world. After that, in Section 6 we investigate the calibration problem of our model.

2 Geometric Lévy process pricing models

We assume that the value process of bond is given by

$$B_t = \exp\{rt\}, \tag{2.1}$$

where r is a positive constant.

A pricing model consists of the following two parts:

(A) The price process S_t of the underlying asset.

(B) The rule to compute the prices of options.

For the part (A) we adopt the geometric Lévy processes, so the part (A) is reduced to the selecting problem of a suitable class of the geometric Lévy processes. For the second part (B) we adopt the martingale measure method, so the part (B) is reduced to the selecting procedure of a suitable martingale measure Q, and then the price of an option X is given by $e^{-rT}E_Q[X]$. Our studies in this paper are carried on under such a framework.

2.1 Geometric Lévy processes

The price process S_t of a stock is assumed to be defined as what follows. We suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t, 0 \geq t \geq T\}$ are given, and that the price process $S_t = S_0 e^{Z_t}$ of a stock is defined on this probability space and given in the form

$$S_t = S_0 e^{Z_t}, \quad 0 \le t \le T, \tag{2.2}$$

where Z_t is a Lévy process. We call such a process S_t the geometric Lévy precess (GLP), and we denote the generating triplet of Z_t by $(\sigma^2, \nu(dx), b)$.

Throughout this paper we assume that $\mathcal{F}_t = \sigma(S_s, 0 \le s \le t) = \sigma(Z_s, 0 \le s \le t)$ and $\mathcal{F} = \mathcal{F}_T$. A probability measure Q on $(\Omega, \mathcal{F})$ is called an equivalent martingale measure of S_t if $Q \sim P$ and $e^{-rt}S_t$ is $(\mathcal{F}_t, Q)$-martingale.

The price process S_t has the following another expression

$$S_t = S_0 \mathcal{E}(\tilde{Z})_t \tag{2.3}$$

where $\mathcal{E}(\tilde{Z})_t$ is the Doléans-Dade exponential (or stochastic exponential) of $\tilde{Z}_t$, and $\tilde{Z}_t$ is a Lévy process corresponding to the original Lévy process Z_t. The generating triplet of $\tilde{Z}_t$, say $(\tilde{\sigma}^2, \tilde{\nu}(dx), \tilde{b})$, is

$$\tilde{\sigma}^2 = \sigma^2 \tag{2.4}$$

$$\tilde{\nu}(dx) = (\nu \circ J^{-1})(dx), \quad J(x) = e^x - 1, \tag{2.5}$$

$$\begin{aligned} \tilde{b} &= b + \frac{1}{2}\sigma^2 + \int_{\{|x| \le 1\}} (e^x - 1 - x)\nu(dx) \\ &\quad + \int_{\{x < -1\}} (e^x - 1)\nu(dx) - \int_{\{\log 2 < x \le 1\}} (e^x - 1)\nu(dx). \end{aligned} \tag{2.6}$$

Remark 1 (i) *It holds that supp* $\tilde{\nu} \subset (-1, \infty)$.
(ii) *If* $\nu(dx)$ *has the density* $n(x)$, *then* $\tilde{\nu}(dx)$ *has the density* $\tilde{n}(x)$ *and* $\tilde{n}(x)$ *is given by*

$$\tilde{n}(x) = \frac{1}{1+x} n(\log(1+x)). \tag{2.7}$$

(iii) S_t *satisfies the following stochastic differential equation*

$$dS_t = S_{t-} d\tilde{Z}_t. \tag{2.8}$$

(iv) *The relations between* Z_t *and* $\tilde{Z}_t$ *are more precisely discussed in* [24]*,where the stochastic logarithm of* X_t, $\mathcal{L}(X)_t$, *is defined and the following relations are obtained.*

$$Z_t = \log \mathcal{E}(\tilde{Z})_t, \quad \tilde{Z}_t = \mathcal{L}(e^{Z_\cdot})_t \tag{2.9}$$

Many candidates for the suitable Lévy process have been proposed. We give some examples below.
(1) Stable process (Mandelbrot, Fama(1963))
(2) Jump diffusion process (Merton(1973))
(3) Variance Gamma process (Madan(1990))
(4) Generalized Hyperbolic process (Eberlein(1995)
(5) CGMY process (Carr-Geman-Madam-Yor(2000))
(6) Normal inverse Gaussian process (Barndorff-Nielsen)
(7) finite moment log stable process(Carr-Wu(2003))

2.2 Equivalent martingale measures

Many candidates for the equivalent martingale measure have been proposed as follows.
(1) Minimal Martingale Measure (MMM) (Föllmer-Schweizer(1991))
(2) Esscher Martingale Measure (ESMM) (Gerber-Shiu(1994), B-D-E-S(1996))
(3) Minimal Entropy Martingale Measure (MEMM) (Miyahara(1996), Frittelli(2000))
(4) Utility Martingale Measure (Utility-MM)

Among those equivalent martingale measures we put our focus on ESMM and MEMM, and investigate the properties of them in section 3.

Remark 2 *Sometimes Mean Correcting Martingale Measure* (MCMM) *is used for jump-diffusion models.*

3 Esscher Martingale Measure (ESMM) and Minimal Entropy Martingale Measure (MEMM)

3.1 Esscher transforms

The Esscher transform is very popular and thought to be very important method in the actuary theory. (See [18]). Esscher has introduced the risk function and the transformed risk function for the calculation of collective risk. His idea has been developed in his work [13] and by many authors, and played very important roles in the option pricing theory.

Definition 1 *Let R be a random variable and h be a constant. Then the probability measure $P^{(ESS)}_{R,h}$ defined by*

$$\frac{dP^{(ESS)}_{R,h}}{dP}|_{\mathcal{F}} = \frac{e^{hR}}{E[e^{hR}]} \tag{3.1}$$

is called the Esscher transformed measure of P by the risk variable R and the index h, and this measure transformation is called the Esscher transform by the risk variable R and the index h.

Definition 2 *Let $R_t, 0 \leq t \leq T$, be a stochastic process. Then the Esscher transformed measure of P by the risk process R_t and the index process h_s is the probability measure $P^{(ESS)}_{R_{[0,T]},h}$ defined by*

$$\frac{dP^{(ESS)}_{R_{[0,T]},h_{[0,T]}}}{dP}|_{\mathcal{F}} = \frac{e^{\int_0^T h_s dR_s}}{E[e^{\int_0^T h_s dR_s}]} \tag{3.2}$$

This measure transformation is called the Esscher transform by the risk process R_t and the index process h_s.

Definition 3 *In the above definitions, if the index index process is chosen so that the $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is a martingale measure of S_t, then $P^{(ESS)}_{R_{[0,T]},h_{[0,T]}}$ is called the Esscher transformed martingale measure of S_t by the risk process R_t, and it is denoted by $P^{(ESS)}_{R_{[0,T]}}$ or simply $P^{(ESS)}_{R.}$.*

3.2 Esscher transformed martingale measures for geometric Lévy processes

3.2.1 Simple return process and compound return process

When we give a certain risk process R_t, we obtain a corresponding Esscher transformed martingale measure if it exists. As we have seen in the previous section, the GLP has two kinds of representation such that

$$S_t = S_0 e^{Z_t} = S_0 \mathcal{E}(\tilde{Z})_t.$$

The processes Z_t and $\tilde{Z}_t$ are candidates for the risk process.

We shall see the economical meaning of them. For this purpose, we will review the discrete time approximation of geometric Lévy processes.

Set

$$S_k^{(n)} = S_{k/2^n}, \quad k = 1, 2, \ldots. \tag{3.3}$$

According to the above two kinds of expression of S_t, we obtain two kinds of approximation formula.

First one is

$$S_k^{(n)} = S_0 e^{Z_k^{(n)}}, \quad k = 1, 2, \ldots, \tag{3.4}$$

where $Z_k^{(n)} = Z_{k/2^n}$.

Second approximation is

$$S_k^{(n)} = S_0 \mathcal{E}(Y^{(n)})_k, \quad k = 1, 2, \ldots, \tag{3.5}$$

where $\mathcal{E}(Y^{(n)})_k$ is the discrete time Doléans-Dade exponential of $Y_k^{(n)}$,

$$\mathcal{E}(Y^{(n)})_k = \prod_{j=1}^{k} \left(1 + (Y_j^{(n)} - Y_{j-1}^{(n)})\right) \tag{3.6}$$

and $Y_k^{(n)}$ is defined from the following relations

$$e^{Z_k^{(n)}} = \mathcal{E}(Y^{(n)})_k = \prod_{j=1}^{k} \left(1 + (Y_j^{(n)} - Y_{j-1}^{(n)})\right), \quad k = 1, 2, \ldots. \tag{3.7}$$

So we obtain

$$e^{\triangle Z_k^{(n)}} = e^{Z_k^{(n)} - Z_{k-1}^{(n)}} = \left(1 + (Y_k^{(n)} - Y_{k-1}^{(n)})\right) = 1 + \triangle Y_k^{(n)}. \tag{3.8}$$

From this we obtain

$$\triangle Y_k^{(n)} = e^{\triangle Z_k^{(n)}} - 1 = \frac{S_k^{(n)}}{S_{k-1}^{(n)}} - 1 = \frac{\triangle S_k^{(n)}}{S_{k-1}^{(n)}} \tag{3.9}$$

and we know that $\triangle Y_k^{(n)}$ is the simple return process of $S_k^{(n)}$.

On the other hand, we obtain from the definition of $\triangle Z_k^{(n)}$

$$\triangle Z_k^{(n)} = \log S_k^{(n)} - \log S_{k-1}^{(n)} = \log \left(1 + \frac{\triangle S_k^{(n)}}{S_{k-1}^{(n)}}\right), \tag{3.10}$$

and we know that $\triangle Z_k^{(n)}$ is the increment of log-returns and it is called the compound return process of $S_k^{(n)}$.

Remark 3 *The terms* 'simple return' *and* 'compound return' *were introduced in* [1, p.294]*, and well known in economics.* (See also [39].)

For $t \in (\frac{k-1}{2^n}, \frac{k}{2^n})$ we define

$$Z_t^{(n)} = Z_k^{(n)}, \quad Y_t^{(n)} = Y_k^{(n)}. \tag{3.11}$$

It is easy to see that the process $Z_t^{(n)}$ converges to the process Z_t when n goes to ∞.

On the other hand the process $Y_t^{(n)}$ converges to the process $\tilde{Z}_t$. As we have seen, S_t satisfies the following stochastic differential equation

$$dS_t = S_{t-} d\tilde{Z}_t. \tag{3.12}$$

From this it follows that

$$d\tilde{Z}_t = \frac{dS_t}{S_{t-}}. \tag{3.13}$$

(For the justification of this formula, see [24].) Comparing the formulae (3.9) and (3.13), we know that the process $Y_t^{(n)}$ is the approximation process in the procedure of solving the equation (3.13) for $\tilde{Z}_t$. This fact means that the process $Y_t^{(n)}$ converges to the process $\tilde{Z}_t$.

Based on the above observation, it is natural for us to give the following definition.

Definition 4 *The process $\tilde{Z}_t$ is called the simple return process of S_t, and the process Z_t is called the compound return process of S_t.*

3.2.2 Two kinds of Esscher transformed martingale measures

Suppose that Z_t is adopted as the risk process. Then, if the corresponding Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$ is well defined, then it should be called the *'compound return Esscher transformed martingale measure'*. This is the Gerber-Shiu's Esscher martingale measure introduced in [18], and the term *'Esscher martingale measure'* is usually suggesting this compound return Esscher transformed martingale measure $P^{(ESS)}_{Z_{[0,T]}}$.

Next we consider the case where $\tilde{Z}_t$ is adopted as the risk process. If the corresponding Esscher transformed martingale measure $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ exists, then it should be called the *'simple return Esscher transformed martingale measure'*.

In [24] the following results have been obtained.

Proposition 1 ([24], Theorem 4.2) *The compound return Esscher transformed martingale measure* $P^{(ESS)}_{Z_{[0,T]}}$ *is unique if it exists.*

Proposition 2 ([24], Theorem 4.5) *The simple return Esscher transformed martingale measure* $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ *is unique if it exists.*

We will see here the relation between the Esscher transform and the minimal entropy martingale measure (MEMM). We first give the definition of the MEMM.

Definition 5 (minimal entropy martingale measure (MEMM)) *If an equivalent martingale measure* P^* *satisfies*

$$H(P^*|P) \leq H(Q|P) \qquad \forall Q: \quad equivalent \quad martingale \quad measure, \tag{3.14}$$

then P^* *is called the minimal entropy martingale measure* (MEMM) *of* S_t*. Where* $H(Q|P)$ *is the relative entropy of* Q *with respect to* P

$$H(Q|P) = \left\{ \begin{array}{ll} \int_\Omega \log[\frac{dQ}{dP}]dQ, & if \quad Q \ll P, \\ \infty, & otherwise, \end{array} \right\}. \tag{3.15}$$

From the proof of [17, Theorem 3.1] it follows that

Proposition 3 *The simple return Esscher transformed martingale measure* $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ *of* S_t *is the minimal entropy martingale measure* (MEMM) *of* S_t*.*

Based on the above results, we give the following definition.

Definition 6 (i) *The compound return Esscher transformed martingale measure* $P^{(ESS)}_{Z_{[0,T]}}$ *is called the 'Esscher martingale measure* (ESMM)*' and denoted by* $P^{(ESMM)}$.
(ii) *The simple return Esscher transformed martingale measure* $P^{(ESS)}_{\tilde{Z}_{[0,T]}}$ *is called the 'minimal entropy martingale measure* (MEMM)*' and denoted by* P^* *(or* $P^{(MEMM)}$*).*

Remark 4 *For the jump-diffusion models, the Brownian motion can be adopted as the risk process. In that case the corresponding Esscher transformed martingale measure is the mean correcting martingale measure.* (See [38] or [7]).

3.3 Existence theorems of ESMM and MEMM for geometric Lévy processes

The uniqueness theorems have been stated in the previous section. We next study the Existence problem of Esscher transformed martingale measures.

3.3.1 Existence theorem of ESMM

We suppose that the expectations which appear in what follows exist. Then the martingale condition for an Esscher transformed probability measure $Q = P^{(ESS)}_{Z_{[0,T]},h}$ is

$$E_Q[e^{-r}S_1] = e^{-r}S_0 E_Q[e^{Z_1}] = e^{-r}S_0 \frac{E_P[e^{(h+1)Z_1}]}{E_P[e^{hZ_1}]} = S_0. \tag{3.16}$$

This condition is equal to the following condition

$$E_P[e^{(h+1)Z_1}] = e^r E_P[e^{hZ_1}], \tag{3.17}$$

and this is also equivalent to the following expression,

$$\phi(-i(h+1)) = e^r \phi(-ih), \tag{3.18}$$

where $\phi(u)$ is the characteristic function of Z_1 ($\phi(u) = E_P[e^{iuZ_1}]$).

To formulate the existence theorem, we set

$$\begin{aligned} f^{(ESMM)}(h) &= b + (\frac{1}{2} + h)\sigma^2 + \int_{\{|x|\leq 1\}} \left((e^x - 1)e^{hx} - x\right) \nu(dx) \\ &\quad + \int_{\{|x|>1\}} (e^x - 1)e^{hx}\, \nu(dx), \end{aligned} \tag{3.19}$$

Then we obtain

Theorem 1 (Existence condition for ESMM) *If the equation*

$$f^{(ESMM)}(h) = r, \tag{3.20}$$

has a solution h^*, then the ESMM of S_t, $P^{(ESMM)}$, exists and

$$P^{(ESMM)} = P^{(ESS)}_{Z_{[0,T]},h^*} = P^{(ESS)}_{Z_T,h^*} \tag{3.21}$$

The process Z_t is also a Lévy process under $P^{(ESMM)}$ and the generating triplet of Z_t under $P^{(ESMM)}$, say $(\sigma^{(ESMM)^2}, \nu^{(ESMM)}(dx), b^{(ESMM)})$, is

$$\sigma^{(ESMM)^2} = \sigma^2, \tag{3.22}$$

$$\nu^{(ESMM)}(dx) = e^{h^*x}\nu(dx), \tag{3.23}$$

$$b^{(ESMM)} = b + h^*\sigma^2 + \int_{\{|x|\leq 1\}} x(e^{h^*x} - 1)\nu(dx). \tag{3.24}$$

(Proof) The equation (5.5) is equivalent to the condition (5.3). Therefore $P^{(ESS)}_{Z_{[0,T]},h^*}$ is a martingale measure of S_t.

The characteristic function of Z_t under $P^{(ESMM)} = P^{(ESS)}_{Z_{[0,T]},h^*}$, $\phi_t^{(ESMM)}(u)$, is by definition

$$\phi_t^{(ESMM)}(u) = E_{P^{(ESMM)}}[e^{iuZ_t}] = \frac{E_P[e^{iuZ_t}e^{h^*Z_T}]}{E_P[e^{h^*Z_T}]}. \tag{3.25}$$

And this is equal to

$$\frac{E_P[e^{(iu+h^*)Z_t}]}{E_P[e^{h^*Z_t}]} = \frac{\phi_t(u - ih^*)}{\phi_t(-ih^*)}. \tag{3.26}$$

By simple calculation we obtain

$$\begin{aligned}\phi_t^{(ESMM)}(u) = \ & \exp\Big\{t\Big(-\tfrac{1}{2}\sigma^2 + i(b + h^*\sigma^2 + \textstyle\int_{\{|x|\leq 1\}} x(e^{h^*x} - 1)\nu(dx))u \\ & + \textstyle\int_{\{|x|\leq 1\}}(e^{iux} - 1 - iux)e^{h^*x}\nu(dx) \\ & + \textstyle\int_{\{|x|>1\}}(e^{iux} - 1)e^{h^*x}\nu(dx)\Big)\Big\}. \end{aligned} \tag{3.27}$$

Using this formula, we can see that the martingale condition (3.18) is reduced to the equation 3.20). (Q.E.D.)

3.3.2 Existence theorem of MEMM

As we have mentioned in the previous section, the MEMM, P^*, is the simple return Esscher transformed martingale measure. ($P^* = P^{(ESS)}_{\tilde{Z}_{[0,T]}}$). The existence theorem of the MEMM is obtained in [17].

Set

$$f^{(MEMM)}(\theta) = b + (\frac{1}{2}+\theta)\sigma^2 + \int_{\{|x|\le 1\}} \left((e^x-1)e^{\theta(e^x-1)} - x\right) \nu(dx) + \int_{\{|x|>1\}} (e^x-1)e^{\theta(e^x-1)} \nu(dx) \quad (3.28)$$

Then the following result is obtained ([17, Theorem 3.1]).

Theorem 2 (Existence condition for MEMM) *If the equation*

$$f^{(MEMM)}(\theta) = r \quad (3.29)$$

has a solution θ^*, then the MEMM of S_t, P^*, exists and

$$P^* = P^{(MEMM)} = P^{(ESS)}_{\tilde{Z}_{[0,T]},\theta^*} = P^{(ESS)}_{\tilde{Z}_T,\theta^*} \quad (3.30)$$

The process Z_t is also a Lévy process under P^* and the generating triplet of Z_t under P^*, say $(\sigma^{*2}, \nu^*(dx), b^*)$, is

$$\sigma^{*2} = \sigma^2, \quad (3.31)$$

$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx), \quad (3.32)$$

$$b^* = b + \theta^*\sigma^2 + \int_{\{|x|\le 1\}} x(e^{\theta^*(e^x-1)} - 1)\nu(dx). \quad (3.33)$$

(Proof) The results of this theorem follows directly from the proof of [17, Theorem 3.1]. (Q.E.D.)

Remark 5 (i) *The function $f^{(MEMM)}(\theta)$ is a non-decreasing function of θ.*
(ii) *If S_t is integrable, then it holds that*

$$E(S_t) = S_0 \exp(t f^{(MEMM)}(0)). \quad (3.34)$$

(iii) *If the condition that $f^{(MEMM)}(0) > r$ is satisfied, then the solution θ^* of (3.29) is negative ($\theta^* < 0$), when it exists. Such cases occur very often.*

3.4 Comparison of ESMM and MEMM

The ESMM and the MEMM are both obtained by Esscher transform, but they have slightly different properties. We will survey the properties and the differences of them.

(1) Corresponding risk process:

The risk process corresponding to the ESMM is the compound return process, and the risk process corresponding to the MEMM is the simple return process. The simple return process seems to be more essential in the relation to the original process rather than the compound return process. In this sense we can say that the MEMM is more reasonable martingale measure than the ESMM.

(2) Existence condition:

As we have seen in the previous section, for the existence of ESMM, $P^{(ESMM)}$, the following condition

$$\int_{\{|x|>1\}} |(e^x - 1)e^{h^* x}| \, \nu(dx) < \infty \tag{3.35}$$

is necessary. On the other hand, for the existence of MEMM, P^*, the corresponding condition is

$$\int_{\{|x|>1\}} |(e^x - 1)e^{\theta^*(e^x-1)}| \, \nu(dx) < \infty. \tag{3.36}$$

This condition is satisfied for wide class of Lévy measures, if $\theta^* < 0$. Namely, the former condition is strictly stronger than the latter condition.

This means that the MEMM may be applied to the wider class of models than the ESMM. This difference does work in the stable process cases. In fact the MEMM method can be applied to the geometric stable model but the ESMM method can not be applied to this same model.

(3) Corresponding utility function:

The ESMM is corresponding to power utility function or logarithm utility function. (See [18, pp.175-177] or [19, Corollary 6.3]). On the other hand the MEMM is corresponding to the exponential utility function. (See [16, §3] or [19, §6.1]).

We remark here that, in the case of ESMM, the power parameter of the utility function depends on the parameter value h^* of the Esscher transform.

We also remark that, in the case of MEMM, the relation of the MEMM to the utility indifference price is known. (See [17, §4]. This result is generalized by C. Stricker [40].)

(4) Properties special to MEMM:

a) Minimal distance to the original probability:

The relative entropy is very popular in the field of information theory, and it is called Kullback-Leibler Information Number(see [21, p.23]) or Kullback-Leibler distance (see [10, p.18]). Therefore we can state that the MEMM is the nearest equivalent martingale measure to the original probability P in the sense of Kullback-Leibler distance. Recently the idea of minimal distance martingale measure is studied. In [19] it is mentioned that the relative entropy is the typical example of the distance in their theory.

b) Large deviation property:

The large deviation theory is closely related to the minimum relative entropy analysis, and the Sanov's theorem or Sanov property is well-known (see, e.g. [10, p.291-304] or [21, p.110-111]). This theorem says that the MEMM is the most possible empirical probability measure of paths of price process in the class of the equivalent martingale measures. In this sense the MEMM should be considered to be the exceptional measure in the class of all equivalent martingale measures.

As the result of the above discussions, we can say that the MEMM has many better properties than the ESMM in the theoretical sense.

4 [GLP & MEMM] Pricing Model

In this section we explain the [GLP & MEMM] pricing model and see examples of the model.

4.1 Model

Now we can define the [GLP & MEMM] Pricing Model, which was first introduced in [30]. The [GLP & MEMM] Pricing Model is such a model: (A) The price process S_t is a geometric Lévy process (GLP).

(B) The price of an option X is defined to be $e^{-rT}E_{P^*}[X]$, where P^* is the MEMM.

Of course this model can be considered for the cases where the MEMM exists.

4.2 Sufficient Conditions for the Existence of the MEMM

The existence problem of the MEMM of geometric Lévy processes has been studied in [29], [6] and [30], and finally those results are generalized in [17] as the following form.

Theorem 3 (Fujiwara-Miyahara [17, Theorem 3.1]) *Suppose that the following condition* **(C)** *holds*
Condition (C) *There exists $\theta^* \in R$ which satisfies the following conditions :*

$$(\mathbf{C})_1 \qquad \int_{\{x>1\}} e^x e^{\theta^*(e^x-1)}\nu(dx) < \infty, \tag{4.1}$$

$$\begin{aligned}(\mathbf{C})_2 \quad & b + (\tfrac{1}{2}+\theta^*)\sigma^2 + \int_{\{|x|>1\}}(e^x-1)e^{\theta^*(e^x-1)}\,\nu(dx) \\ & + \int_{\{|x|\le 1\}}\left((e^x-1)e^{\theta^*(e^x-1)} - x\right)\,\nu(dx) = r. \end{aligned} \tag{4.2}$$

Then the probability measure P^* is well defined and it holds that
(i)(MEMM): *P^* is the MEMM of S_t.*
(ii)(Lévy process): *Z_t is also a Lévy process w.r.t. P^*, and the generating triplet (A^*, ν^*, b^*) of Z_t under P^* is*

$$A^* = \sigma^2, \tag{4.3}$$

$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx), \tag{4.4}$$

$$b^* = b + \theta^*\sigma^2 + \int_{R\setminus\{0\}} xI_{\{|x|\le 1\}}d(\nu^*-\nu). \tag{4.5}$$

4.3 Examples of [GLP & MEMM] Pricing Model

In this section we see several examples of [GLP & MEMM] Pricing Models. To do this, we have to check the existence of the MEMM, i.e. we have to examine that the given geometric Lévy process $S_t = S_0 \exp Z_t$ satisfies the Condition (C). Set

$$\begin{aligned} f(\theta) &= b + (\frac{1}{2} + \theta)\sigma^2 + \int_{\{|x|>1\}} (e^x - 1)e^{\theta(e^x-1)}\,\nu(dx) \\ &\quad + \int_{\{|x|\leq 1\}} \Big((e^x - 1)e^{\theta(e^x-1)} - x\Big)\,\nu(dx). \end{aligned} \tag{4.6}$$

Then the condition $(\mathrm{C})_2$ is equivalent to that θ^* is the solution of

$$f(\theta) = r. \tag{4.7}$$

4.3.1 Geometric Variance Gamma Model

The Lévy measure of Variance Gamma process is of the following form (see [25]).

$$\nu(dx) = C\left(I_{\{x<0\}} \exp(-c_1|x|) + I_{\{x>0\}} \exp(-c_2|x|)\right)|x|^{-1}dx, \tag{4.8}$$

where C, c_1, c_2 are positive constants.

The following results are obtained (see[17] or [34]).

Proposition 4 (1) *If $c_2 \leq 1$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is negative.*
(2) *If $c_2 > 1$ and $f(0) \geq r$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.*
(3) *If $c_2 > 1$ and $f(0) < r$, then the equation $f(\theta) = r$ has no solution.*

4.3.2 Geometric CGMY Model

The Lévy measure of the CGMY process is

$$\nu(dx) = C\left(I_{\{x<0\}} \exp(-G|x|) + I_{\{x>0\}} \exp(-M|x|)\right)|x|^{-(1+Y)}dx, \tag{4.9}$$

where $C > 0, G \geq 0, M \geq 0, Y < 2$ (see [2]). If $Y \leq 0$, then $G > 0$ and $M > 0$ are assumed. We mention here that the case $Y = 0$ is the VG process case, and the case $G = M = 0$ and $0 < Y < 2$ is the symmetric stable process case. In the sequel we assume that $G, M > 0$.

For this model the following results are obtained (see [34]).

Proposition 5 (1) *If $M \leq 1$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is negative.*
(2) *If $M > 1$ and $f(0) \geq r$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.*
(3) *If $M > 1$ and $f(0) < r$, then the equation $f(\theta) = r$ has no solution.*

4.3.3 Geometric Stable Model

We consider the stable model. Suppose that Z_t is a stable process and let $(0, \nu(dx), b)$ be its generating triplet. The Lévy measure is

$$\nu(dx) = c_1 I_{\{x<0\}} |x|^{-(\alpha+1)} dx + c_2 I_{\{x>0\}} |x|^{-(\alpha+1)} dx, \tag{4.10}$$

where $0 < \alpha < 2$ and we assume that

$$c_1 \geq 0, \quad c_2 \geq 0, \quad c = c_1 + c_2 > 0. \tag{4.11}$$

Proposition 6 *Under the assumption $c_1, c_2 > 0$, the equation $f(\theta) = r$ has a unique solution θ^*, and the solution θ^* is negative.*

Remark 6 *Consider the case where $c_1, c_2 > 0$. Under the original measure P, $S_t, t > 0$ is not integrable. But under the MEMM P^*, any moments $E_{P^*}[|S_t|^k], k = 1, 2, \ldots,$ of S_t are finite. This fact follows easily from the result that θ^* is negative, and this property is very useful for the study of option pricing of this model.*

4.4 Option Pricing and Volatility Smile/Smirk Properties

In order to apply the [GLP & MEMM] Pricing Models to the financial problems, we have to establish the methods to compute the option prices. Namely we have to compute the expectations $E_{P^*}[F(\omega)]$, where $F(\omega)$ is a functional of Lévy process.

4.4.1 European Type Options

If a contingent claim C is depending only on the terminal value of the stock price $S_T = S_0 e^{Z_T}$, then we can compute the price of C as what follows.

Let $C = f(S_T) = f(S_0 e^{Z_T}) = F(Z_T)$, $(F(z) = f(S_0 e^z))$, and set $C(t,y) = E_{P^*}[e^{-r(T-t)} f(S_T)|S_t = y]$ and $\tilde{C}(t,z) = E_{P^*}[e^{-r(T-t)} F(Z_T)|Z_t = z] = E_{P^*}[e^{-r(T-t)} f(S_T)|S_t = S_0 e^z]$. (Remark that $\tilde{C}(t,z) = C(t, S_0 e^z)$.) Since the process Z_t is a Lévy process with the generating triplet $(\sigma^2, \nu^*(dx), b^*)$, $\tilde{C}(t,z)$ satisfies the following equation under the assumption of the smoothness of $\tilde{C}(t,z)$.

$$
\begin{aligned}
-\frac{\partial \tilde{C}(t,z)}{\partial t} &= \frac{1}{2}\sigma^2 \frac{\partial^2 \tilde{C}(t,z)}{\partial z^2} + b\frac{\partial \tilde{C}(t,z)}{\partial z} \\
&\quad + \int_{-\infty}^{\infty} \left(\tilde{C}(t, z+\tilde{z}) - \tilde{C}(t,z) - \tilde{z}\frac{\partial \tilde{C}(t,z)}{\partial z} 1_{\{|\tilde{z}|<1\}}(\tilde{z}) \right) \nu(d\tilde{z}) \\
&\quad - r\tilde{C}(t,z), \quad 0 \le t < T, \qquad (4.12) \\
\tilde{C}(T,z) &= F(z). \qquad (4.13)
\end{aligned}
$$

Solving this equation, we obtain the option price $C(0, S_0) = \tilde{C}(0,0)$.

4.4.2 FFT Method for European Call Options

The fast Fourier transform method (FFT method) is very useful for the computation of option prices. We need to compute the such an expectation $E_{P^*}[F(\omega)]$, and in the case of European type options such type of expectations $E_{P^*}[G(S_T)]$. If we know the distribution function $p_T^*(z)$ of Z_T under P^*, then $E_{P^*}[G(S_T)] = \int_{-\infty}^{\infty} G(z) p_T^*(z) dz$. Lévy process is characterized by the generating triplet, and the generating triplet is given explicitly in the characteristic function. So we can assume that the characteristic function $\phi_T^*(u)$ of Z_T under P^* is given and the density function $p_T^*(z)$ is obtained as the inverse Fourier transform of $\phi_T^*(u)$.

For the computer simulation of the theoretical prices of European call options, the FFT method is very useful. Carr and Madan have introduced their idea in [3], and their method has been improved by Cont and Tankov in [8]. We rearrange their ideas in such a form that we can easily apply the formula to our [GLP & MEMM] pricing models.

The characteristic function $\phi_t^*(u)$ of Z_t under the MEMM P^* is

$$\phi_t^*(u) = \phi_{Z_t}^*(u) = E_{P^*}[e^{iuZ_t}] = \exp(\psi_t^*(u)) = \exp(t\psi^*(u)), \quad i = \sqrt{-1}, \tag{4.14}$$

where $\psi^*(u) = \psi_1^*(u)$. Let $\mu_t^*(dz)$ be the distribution of Z_t under the MEMM P^*, and assume that $\mu_t^*(dz) = p_t^*(z)dz$. Then

$$\phi_t^*(u) = \phi_{Z_t}^*(u) = E_{P^*}[e^{iuZ_t}] = \int_{-\infty}^{\infty} e^{iuz} p_t^*(z)dz, \tag{4.15}$$

The price of European call option is

$$C(S_0, K, T) = e^{-rT} E_{P^*}[(S_T - K)^+] = e^{-rT} \int_{-\infty}^{\infty} (S_0 e^z - K)^+ p_T^*(z)dz. \tag{4.16}$$

Set $K/S_0 = e^k$, and define $c(k; S_0, T) = C(S_0, S_0e^k, T)$. Then using (5.3)

$$c(k; S_0, T) = S_0 e^{-rT} \int_{-\infty}^{\infty} (e^z - e^k)^+ p_T^*(z)dz \tag{4.17}$$

We introduce the so-called time value of option

$$\tilde{c}(k; S_0, T) = c(k; S_0, T) - (S_0 - e^{-rT}K)^+ = c(k; S_0, T) - S_0(1 - e^{k-rT})^+ \tag{4.18}$$

and let $\zeta(v; S_0, T)$ be the Fourier transform of $\tilde{c}(k; S_0, T)$

$$\zeta(v; S_0, T) = \int_{-\infty}^{\infty} e^{ivk} \tilde{c}(k; S_0, T)dk. \tag{4.19}$$

Using (5.5)

$$\zeta(v; S_0, T) = S_0 e^{-rT} \int_{-\infty}^{\infty} p_T^*(z)dz \int_z^{rT} e^{ivk}(e^k - e^z)dk. \tag{4.20}$$

and

$$\zeta(v; S_0, T) = S_0 \frac{e^{-rT}\phi_T^*(v - i) - e^{ivrT}}{iv(1 + iv)}. \tag{4.21}$$

The characteristic function $\phi_T^*(u)$ is computed directly from the generating triplet $(\sigma^2, b^*, \nu^*(dx))$, so $\zeta(v; S_0, T)$ is obtained from the above formula. Next, by (), $\tilde{c}(k; S_0, T)$ is obtained by the inverse Fourier transform

$$\tilde{c}(k; S_0, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikv} \zeta(v; S_0, T)dv \tag{4.22}$$

and

$$c(k; S_0, T) = \tilde{c}(k; S_0, T) + (S_0 - e^{-rT}K)^+, \quad K = S_0 e^k. \tag{4.23}$$

Finally we obtain the price of the European call option $C(S_0, K, T)$ as

$$C(S_0, K, T) = c(\log(K/S_0); S_0, T) = \tilde{c}(\log(K/S_0); S_0, T) + (S_0 - e^{-rT}K)^+. \tag{4.24}$$

4.4.3 Volatility Smile/Smirk Properties

The volatility smile/smirk properties are reported for many market prices of options. This fact tells us that the Black-Scholes model is not necessarily best model, and that we should study other models which may have the volatility simile/smirk properties. It is known that the [GLP & MEMM] models have those properties. (See [33]).

5 Physical World and MEMM World

The behavior of the price process S_t is governed by the original probability P, and the movement of S_t is observable. This is the real world (=Physical world).

On the other hand the price of an option X is computed as the expectation $e^{-rT}E_{P^*}[X]$, namely the process S_t is supposed to obey the MEMM P^*. This world is differ from the real world, and this world should be called the imaginary world (=MEMM world).

5.1 From Physical World to MEMM World

Suppose that the price process $S_t = S_0 e^{Z_t}$ is given and the generating triplet of Z_t is (σ^2, ν, b). Let θ^* is the solution of $f(\theta) = r$, where the function $f(\theta)$ is defined by (4.6). Then, by Theorem 3 in §4.2, the generating triplet $(\sigma^{*2}, \nu^*, b^*)$ of Z_t under P^* is

$$\begin{aligned} \sigma^{*2} &= \sigma^2, & (5.1) \\ \nu^*(dx) &= e^{\theta^*(e^x-1)}\nu(dx), & (5.2) \\ b^* &= b + \theta^*\sigma^2 + \int_{\{|x|\le 1\}} xd(\nu^* - \nu) & (5.3) \\ &= b + \theta^*\sigma^2 + \int_{\{|x|\le 1\}} x\left(e^{\theta^*(e^x-1)} - 1\right)\nu(dx). & (5.4) \end{aligned}$$

This triplet determines the prices of options in the framework of [GLP & MEMM] pricing model.

Remark 7 *The condition* $(C)_2$ *for* θ^* *(i.e.* θ^* *is the solution of* $f(\theta) = r$*) is equivalent to the following condition (M)*

$$(M) \quad b^* - r + \frac{1}{2}\sigma^{*2} + \int_{\{|x|\leq 1\}} (e^x - 1 - x)\nu^*(dx) + \int_{\{|x|>1\}} (e^x - 1)\nu^*(dx) = 0. \tag{5.5}$$

We should notice that the θ^* *does not appear explicitly in this formula, and that this formula is just the same condition that* P^* *is a martingale measure of the price process* S_t*.*

Concerning to the martingale condition for more general cases of semimartingales, see [39] *where the triplet's notation for semimartingale are* (B, C, ν)*. (See also* [22].)

5.2 From MEMM World to Physical World

We study the inverse problem of the previous subsection. Suppose that the generating triplet $(\sigma^{*2}, \nu^*, b^*)$ of Z_t under P^* is given. Since we assume that P^* is martingale measure, the condition (M) is satisfied.

We try to construct a probability $\tilde{P}$ such that under $\tilde{P}$ the price process $S_t = S_0 e^{Z_t}$ is geometric Lévy process and the MEMM of $S_t = S_0 e^{Z_t}$ with $\tilde{P}$ is P^*.

Let θ^* be any real number (it is usually supposed that $\theta^* < 0$) and set

$$\tilde{\sigma}^2_{\theta^*} = \sigma^{*2} \tag{5.6}$$

$$\tilde{\nu}_{\theta^*}(x) = e^{-\theta^*(e^x-1)}\nu^*(dx) \tag{5.7}$$

$$\tilde{b}_{\theta^*} = b^* - \theta^*\sigma^2 + \int_{\{|x|\leq 1\}} x\left(e^{-\theta^*(e^x-1)} - 1\right)\nu^*(dx), \tag{5.8}$$

where we assume that all integrals are converge. Then suppose that we could construct the probability measure $\tilde{P}_{\theta^*}$ such that under $\tilde{P}_{\theta^*}$ the process Z_t is a Lévy process with the generating triplet $(\tilde{\sigma}^2_{\theta^*}, \tilde{\nu}_{\theta^*}, \tilde{b}_{\theta^*})$.

It is easy to see that P^* is the MEMM of $S_t = S_0 e^{Z_t}$ with $\tilde{P}_{\theta^*}$. We remark here that there are many geometric Lévy processes whose MEMM is just the same P^*.

5.3 Example: Geometric Stable Process Case

• Parameters in the physical world: (α, c_1, c_2, b), $0 < \alpha < 2$, $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, $-\infty < b < \infty$.

The triplet is $(0, \nu, b)$, and

$$\nu(dx) = \frac{c_1 I_{\{x<0\}}(x) + c_2 I_{\{x>0\}}(x)}{|x|^{(\alpha+1)}} dx.$$

• Parameters in the MEMM world: $(\theta^*, \alpha^*, c_1^*, c_2^*, b^*)$, $\theta^* < 0$, $0 < \alpha^* < 2$, $c_1^*, c_2^* \geq 0$, $c_1^* + c_2^* > 0$, $-\infty < b^* < \infty$, where

$$\nu^*(dx) = e^{\theta^*(e^x-1)} \frac{c_1^* I_{\{x<0\}}(x) + c_2^* I_{\{x>0\}}(x)}{|x|^{(\alpha^*+1)}} dx.$$

and the following martingale condition

$$\begin{aligned} Condition(M): \qquad & b^* + \int_{\{|x|\leq 1\}} (e^x - 1 - x)\nu^*(dx) \\ + & \int_{\{|x|>1\}} (e^x - 1)\nu^*(dx) = r \end{aligned}$$

must be satisfied. So, if we have given the values of $(\theta^*, \alpha^*, c_1^*, c_2^*,)$, then the value of b^* is determined by the above condition (M).

6 Calibration of [GLP & MEMM] pricing model

Suppose that the sequential data of the price process S_t of underlying asset and the data of market prices of options. From these data, we have to select a model which is most fitting to the given data. This is the calibration problem. This problem shall be discussed in section 6.

6.1 Calibration problems

• Given data: the sequential data of the price process S_t, and the data of market prices of options.
• Select the most fitting model to the given data.

To solve this problem, we have to take the following steps.

1) Estimation the price process of the underlying asset in the physical world from the sequential data of it.

2) Calculation of the MEMM from the estimated parameter, and computation of the theoretical prices of options in the estimated MEMM world.

3) Analyzing the fitness of the theoretical prices to the market prices.

4) Determination of the most reasonable model.

6.2 Estimation of Lévy Processes in the Physical World

Usually this procedure is carried on under the restriction of the class of Lévy processes, for example the stable process class, VG process class, etc. Therefore the estimation problem of the process is reduced to the parameter estimation problems.

There are many papers on this subject (see [31] for example).

Denote the estimated probability by $\widehat{P}$, or equivalently, the estimated generating triplet by $(\widehat{\sigma}^2, \widehat{\nu}, \widehat{b}))$.

6.3 Calculation of the theoretical option prices

Let $\widehat{P}$ be the estimated probability in the physical world, and let $\widehat{P}^*$ be the corresponding MEMM. Then the theoretical price of option C is $E_{\widehat{P}^*}[Ce^{-rT}]$. We denote this value by $\widehat{C}^*$.

6.4 Fitness analysis of the estimated model

Suppose that the data, $\eta_l, l = 1, 2, \ldots, L$, of market prices of options C_l. Then we define the fitting error of the model by

$$\epsilon^* = \frac{1}{L} \sum_l \frac{|\widehat{C_l}^* - \eta_l|}{\eta_l}$$

6.5 Determination of the most fitting model

As the results of the above procedure, if the value ϵ^* is small then the fitness of the model to the data is good.

The value ϵ^* depends on the model, namely the selected class of the process. We can conclude that the class whose fitness error is the smallest is the best model.

6.6 Diagram of the Procedure for the Calibration

The path space $\mathcal{D}[0,T]$ is fixed.

Physical World		MEMM World
	$S_t = S_0 e^{Z_t}$	
(σ^2, ν, b) under P	Z_t	$(\sigma^{*2}, \nu^*, b^*)$ under P^*

Data: $\{\xi_j\}$ (time series data)
Estimated: $\widehat{P}$
$(\widehat{\sigma}^2, \widehat{\nu}, \widehat{b})$

$\widehat{\theta}^*$

Transformed: $\widehat{P}^*$
$(\widehat{\sigma}^{*^2}, \widehat{\nu}^*, \widehat{b}^*)$
Theoretical prices: $\widehat{C_l}^*$
(European Call Options)

Data: $\{\eta_l\}$ (European Call Options)

$$\epsilon^* = \frac{1}{L}\sum_l \frac{|\widehat{C_l}^* - \eta_l|}{\eta_l}$$

6.7 Example: Geometric Stable Process Case

Parameter in the physical world

$$(\alpha, c_1, c_2, b)$$

$$\nu(dx) = \frac{c_1 I_{\{x<0\}}(x) + c_2 I_{\{x>0\}}(x)}{|x|^{(\alpha+1)}} dx.$$

Estimators $(\widehat{\alpha}, \widehat{c}_1, \widehat{c}_2, \widehat{b})$

$$\widehat{\nu}(dx) = \frac{\widehat{c}_1 I_{\{x<0\}}(x) + \widehat{c}_2 I_{\{x>0\}}(x)}{|x|^{(\widehat{\alpha}+1)}} dx.$$

$\widehat{\theta}^*$ is determined by

$$\begin{aligned}\widehat{b} \quad &+ \quad \int_{\{|x|>1\}} (e^x - 1)e^{\widehat{\theta}^*(e^x-1)}\,\widehat{\nu}(dx) \\ &+ \quad \int_{\{|x|\le 1\}} \left((e^x - 1)e^{\widehat{\theta}^*(e^x-1)} - x\right)\widehat{\nu}(dx) = r. \qquad (6.1)\end{aligned}$$

The process which determines the theoretical option prices is

$$\widehat{\nu}^*(dx) = e^{\widehat{\theta}^*(e^x-1)}\frac{\widehat{c}_1 I_{\{x<0\}}(x) + \widehat{c}_2 I_{\{x>0\}}(x)}{|x|^{(\widehat{\alpha}+1)}}dx.$$

$$\widehat{b}^* = \widehat{b} + \int_{R\setminus\{0\}} x I_{\{|x|\le 1\}} d(\widehat{\nu}^* - \widehat{\nu}).$$

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