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ABSTRACT

We consider the incomplete assets markets. Then there are many equivalent martingale measures. Among them, the probability measure, which minimizes the relative entropy with respect to the original probability measure $P$, has a special meaning. We call such a measure the canonical martingale measure. The canonical martingale measure is, if exists, unique. We investigate the existence problem of canonical martingale measures.

## 1. Introduction

In the theory of the financial market, equivalent martingale measures are often discussed (Harrison and Pliska 1981, 1983). Suppose that the price process of stocks is the stochastic process $S = (S(t), t \geq 0)$ defined on some probability space $(\Omega, \mathcal{F}, P)$. Under the assumption of the absence of arbitrage opportunities, there is an equivalent probability measure $Q \sim P$ such that $S$ is a martingale under $Q$.(We call such a measure $Q$ the equivalent $S$-martingale measure.)

If the market is complete, the equivalent martingale measure $Q$ is determined uniquely. But, if the market is incomplete, there are many equivalent martingale measures. In such a case, the problem to analyze the mechanism of determining the prices of contingent claims remains to be an open problem. An answer to the above problem is adopting the minimization principle of relative entropy as the criterion of reasonable martingale measure.

The equivalent $S$-martingale measure $P^*$, which has the minimum relative entropy w.r.t. $P$, is called the canonical martingale measure of the price process $S$ (Definition 2). It may be natural to think that the theoretical price (= value) of a contingent claim $X$ is equal to the expectation $E_{P^*}[X]$ of $X$ with respect to $P^*$, in regard of Sanov's Lemma of the large deviation theory. The canonical martingale measure of the price process $S$ is unique, if it exists (Remark 1). Therefore, the existence of canonical martingale measures is the problem to be investigated.

Our first main result is Theorem 3 in §2, which asserts that if the price process $S$ is bounded, then there exists the unique canonical martingale measure. In §3 we investigate the case where the price process $S$ is given by stochastic differential equations(S.D.E.). The second main theorem is Theorem 4, which asserts that in the SDE case there exists the unique canonical martingale measure $P^*$, and that the measure $P^*$ is given by Girsanov's transformation of measure from the original measure $P$. Our results are obtained without the assumption of square integrability of Radon-Nikodym derivatives of $S$ martingale measures with respect to the original measure $P$ (see Remark 4).

## 2. Preliminaries and General Results

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{\mathcal{F}_t, t \geq 0\}$ be an increasing family of sub $\sigma$-fields of $\mathcal{F}$, where we assume that $\mathcal{F}_t$ is right continuous and includes all $P$-negligible sets in $\mathcal{F}$. Suppose that the price process $S = (S(t), t \geq 0)$, $S(t) = (S_1(t), S_2(t), \ldots, S_K(t)) \in R^K$, is given as a K-dimensional $\mathcal{F}_t$-adapted stochastic process defined on $(\Omega, \mathcal{F}, P)$.

Under the above framework, we define $\mathcal{P}(S)$ as the set of all equivalent $S$-martingale measures, namely the set of all probability $Q$ on $(\Omega, \mathcal{F})$ such that $(S(t), t \geq 0)$ is $(\mathcal{F}_t, Q)$-martingale and $Q \sim P$ (absolutely continuous with each other), and we define $\mathcal{M}(S)$ as the set of all $S$-martingale measures which is absolutely continuous with respect to $P$. From the definitions of $\mathcal{P}(S)$ and $\mathcal{M}(S)$, the following lemma is obtained easily.

**Lemma 1** *$\mathcal{P}(S)$ and $\mathcal{M}(S)$ are convex subsets of $\mathcal{M}_1$(=the set of all probability measures on $(\Omega, F)$).*

We denote the relative entropy of $Q$ with respect to $P$ by $H(Q|P)$, and the variation distance of $Q_1$ and $Q_2$ by $\|Q_1 - Q_2\|_{var}$.

**Definition 1** *For the convex subset $\mathcal{E}$ of $\mathcal{M}_1$, we set $H(\mathcal{E}|P) = \inf_{Q \in \mathcal{E}} H(Q|P)$. If this infimum is attained at a point in $\mathcal{E}$, then the point is called minimum point in $\mathcal{E}$.*

**Remark 1** *If a minimum point exists, then it is unique. This fact follows from the strict convexity of the relative entropy.*

The basic properties of the relative entropy are described in the following lemmas.

**Lemma 2 (Csiszár 1984)** *Let $\mathcal{E}$ be a convex subset of $\mathcal{M}_1$, and let $Q_n, n = 1, 2, \ldots,$ be a sequence from $\mathcal{E}$ such that $\lim_{n\to\infty} H(Q_n|P) = H(\mathcal{E}|P)$. Then*

*(1) there is a probability measure $Q_\infty$ such that $lim_{n\to\infty}\|Q_n - Q_\infty\|_{var} = 0$.*

*(2) $H(Q_\infty|P) \leq H(\mathcal{E}|P)$.*

**Lemma 3 (Csiszár 1975, Lemma 2.1)** *Let $Q_0, Q_1 \ll P$ and set $Q_\alpha = \alpha Q_1 + (1-\alpha)Q_0$ for $\alpha, 0 < \alpha < 1$. Then*

*(1) $H(Q_\alpha|P) \geq H(Q_0|P), \forall \alpha, 0 < \alpha < 1 \Longleftrightarrow H(Q_1|P) \geq H(Q_1|Q_0) + H(Q_0|P)$.*

*(2)$\exists \alpha_0, 0 < \alpha_0 < 1$, such that $H(Q_\alpha|P) \geq H(Q_{\alpha_0}|P), \quad \forall \alpha, 0 < \alpha < 1$*

$$\Longleftrightarrow H(Q_\alpha|P) = H(Q_\alpha|Q_{\alpha_0}) + H(Q_{\alpha_0}|P), \quad \forall \alpha, 0 \leq \alpha \leq 1.$$

**Lemma 4** *Suppose that $Q_0, Q_1 \ll P$, $Q_1 \sim P$ , and $H(Q_1|P) < \infty$. And set $Q_\alpha = \alpha Q_1 + (1-\alpha)Q_0$ for $\alpha, 0 \leq \alpha \leq 1$.Then*

*(1) there exist a unique $\alpha_0, 0 \leq \alpha_0 \leq 1$, such that $H(Q_{\alpha_0}|P) = \inf_{0\leq\alpha\leq 1} H(Q_\alpha|P)$.*

*(2) $Q_{\alpha_0}$ satisfies the condition that $Q_{\alpha_0} \sim P$.*

(Proof) Since the set $\{Q_\alpha, 0 \le \alpha \le 1\}$ is a convex closed subset of $\mathcal{M}_1$, the result (1) follows from Lemma 2.

The proof of (2) is as follows.

Case 1. If $\alpha_0 = 1$, then (2) is trivial.

Case 2. If $\alpha_0 = 0$, then by Lemma 3 (1) it holds that $\infty > H(Q_1|P) \ge H(Q_1|Q_0) + H(Q_0|P)$. Therefore it follows that $H(Q_1|Q_0) < \infty$ and so $Q_1 \ll Q_0$. Using the assumptions that $Q_1 \sim P$ and $Q_0 \ll P$ , we know that $Q_0 \sim P$.

Case 3. If $0 < \alpha_0 < 1$, then by Lemma 3 (2), $\infty > H(Q_1|P) = H(Q_1|Q_{\alpha_0}) + H(Q_{\alpha_0}|P)$. So we obtain the result in the same way as case 2. (Q.E.D.)

Using the above lemmas, we can prove the following theorems.

**Theorem 1** *If $H(\mathcal{P}(S)|P) < \infty$, then $H(\mathcal{P}(S)|P) = H(\mathcal{M}(S)|P)$.*

(Proof) From the definitions of $\mathcal{P}(S)$ and $\mathcal{M}(S)$ , the inequality $H(\mathcal{P}(S)|P) \ge H(\mathcal{M}(S)|P)$ is obvious.

Let $\{Q_n, n = 1, 2, ...\}$ be a sequence in $\mathcal{M}(S)$ such that $H(Q_n|P) \downarrow H(\mathcal{M}(S)|P)$, and let $\tilde{Q}$ be a point of $\mathcal{P}(S)$ such that $H(\tilde{Q}|P) < \infty$. Then we can apply Lemma 4 to $Q_n$ and $\tilde{Q}$, and we know that there is a point $Q'_n$ on the segment $\overline{Q_n\tilde{Q}}$ such that $Q'_n \sim P$ and $H(Q'_n|P) \le H(Q_n|P)$.

Since $\mathcal{M}(S)$ is convex, it follows that $Q'_n \in \mathcal{M}(S)$, and from $Q'_n \sim P$ it follows that $Q'_n \in \mathcal{P}(S)$. Therefore we have proved that

$$H(\mathcal{P}(S)|P) \le \liminf_{n\to\infty} H(Q'_n|P) \le \liminf_{n\to\infty} H(Q_n|P) = H(\mathcal{M}(s)|P)$$

and the proof is completed.(Q.E.D.)

**Corollary 1** *If $H(\mathcal{P}(S)|P) < \infty$ and $\mathcal{P}(S)$ has the minimum point, then that point is also the minimum point of $\mathcal{M}(S)$.*

**Theorem 2** *Suppose that $H(\mathcal{P}(S)|P) < \infty$. If there exists the minimum point in $\mathcal{M}(S)$, then the point is also the minimum point in $\mathcal{P}(S)$.*

(Proof) Let $\hat{Q}$ be the minimum point in $\mathcal{M}(S)$, and let $\tilde{Q}$ be a point in $\mathcal{P}(S)$ such that $H(\tilde{Q}|P) < \infty$. Applying Lemma 4 to the segment $\overline{\hat{Q}\tilde{Q}}$, we know that there exists a point $\hat{Q}'$ in $\overline{\hat{Q}\tilde{Q}}$ such that $H(\hat{Q}'|P) \le H(\hat{Q}|P)$ and $\hat{Q}' \sim P$. On the other hand, from the convexity of $\mathcal{M}(S)$ it follows that $\hat{Q}' \in \mathcal{M}(S)$. Thus we have obtained the results that $\hat{Q}' \in \mathcal{P}(S)$ and

$$H(\hat{Q}'|P) \le H(\hat{Q}|P) = H(\mathcal{M}(S)|P) \le H(\mathcal{P}(S)|P).$$

From the uniqueness of the minimum point, it follows that $\hat{Q} = \hat{Q}'$, and the proof is completed. (Q.E.D.)

**Definition 2** *If the ninimum point in $\mathcal{P}(S)$ exists, then that point is called the canonical martingale measure of $S$.*

**Theorem 3** *Assume that the price process $S$ is bounded and assume that $H(\mathcal{P}(S)|P) < \infty$. Then the canonical martingale measure $P^*$ exists and is unique.*

(Proof) If we prove that $\mathcal{M}(S)$ has the minimum point, the result of theorem follows from Theorem 2. Let $Q_n \in \mathcal{M}(S)$ be a sequence such that $H(Q_n|P) \downarrow H(\mathcal{M}(S)|P)$, and put $q_n(\omega) = \frac{dQ_n}{dP}(\omega)$. From Lemma 2 it follows that there exists a probability measure $\hat{Q}$ such that $Q_n \to \hat{Q}$ in variation ($\|Q_n - \hat{Q}\|_{var} \to 0$), and

$$H(\hat{Q}|P) \leq \liminf_{n\to\infty} H(Q_n|P) = H(\mathcal{M}(S)|P)$$

It is well-known (see Ihara 1993, Theorem 1.5.3) that

$$\|Q_m - Q_n\|_{var} = \int_\Omega |q_m - q_n| dP(\omega)$$

So it follows that $\hat{Q} \ll P$ and

$$\|Q_n - \hat{Q}\|_{var} = \int_\Omega |q_n - \hat{q}| dP(\omega) \to 0, \quad where \quad \hat{q} = \frac{d\hat{Q}}{dP}.$$

We shall show next that $\hat{Q} \in \mathcal{M}(S)$. Since $Q_n \in \mathcal{M}(S)$, it holds that for any $\mathcal{F}_t$-measurable bounded function $g(\omega)$ and for $s > t$

$$\begin{aligned} E_{Q_n}[S_s g] &= E_{Q_n}[E_{Q_n}[S_s g|\mathcal{F}_t]] = E_{Q_n}[E_{Q_n}[S_s|\mathcal{F}_s]g] = E_{Q_n}[S_t g] \\ &= \int_\Omega S_t g q_n dP \to \int_\Omega S_t g \hat{q} dP = E_{\hat{Q}}[S_t g] \quad (n \to \infty) \end{aligned}$$

where we use the assumption of boundedness of $S_t$. On the other hand it is obvious that

$$E_{Q_n}[S_s g] = \int_\Omega S_s g q_n dP \to \int_\Omega S_s g \hat{q} dP = E_{\hat{Q}}[S_s g] \quad (n \to \infty)$$

Therefore $E_{\hat{Q}}[S_s g] = E_{\hat{Q}}[S_t g]$ for any bounded $\mathcal{F}_t$ -measurable function $g(\omega)$. This proves that $\hat{Q}$ is a $S$-martingale measure. The fact that $\hat{Q} \in \mathcal{P}(M)$ follows from Theorem 2, and the proof is complete. (Q.E.D.)

By Theorem 3, the existence of the canonical martingale measure in the case that the price process is bounded is assured. This may be enough in the practical point of view, but is not enough in the theoretical point of view. In the next section we investigate a case where the price process is not bounded.

## 3. The Case of Stochastic Differential Equation

In this section we investigate the case where the price process $S_t$ is given by stochastic differential equations(SDE). Let $W(t) = (W_1(t), \ldots, W_d(t))$ be the $d$-dimensional $(\mathcal{F}_t, P)$ Wiener process, and assume that $\mathcal{F}_t = \mathcal{F}_t^W = \sigma\{W(s), 0 \leq s \leq t\}$ and $\mathcal{F} = \mathcal{F}_\infty^W$. Let the price process $S(t) = (S_1(t), \ldots, S_K(t))$ satisfy the following stochastic differential equations

$$dS_k(t) = b_k(t, S(t))dt + \sum_{j=1}^{d} a_{kj}(t, S(t))dW_j(t), \quad k = 1, \ldots, K \tag{1}$$

or equivalently the following stochastic integral equations

$$S_k(t) = S_k(0) + \int_0^t b_k(s, S(s))ds + \sum_{j=1}^{d} \int_0^t a_{kj}(s, S(s))dW_j(s), \quad k = 1, \ldots, K \qquad (2)$$

where we assume that $b_k$ and $a_{kj}, k = 1, 2, \ldots, K, j = 1, 2, \ldots, d$, satisfy the global Lipschitz condition, so that the above equations have a unique solution.

**Theorem 4** *Suppose that the price process $S(t) = (S_1(t), \ldots, S_K(t))$ is given by (1) or (2) and assume that $H(\mathcal{P}(S)|P) < \infty$ .Then there exists the canonical martingale measure $P^*$, i.e. $P^*$ is the minimum point of $\mathcal{P}(S)$, and $P^*$ is obtained by Girsanov transformation from $P$. The canonical martingale measure $P^*$ is also the minimum point of $\mathcal{M}(S)$.*

For the proof of this theorem, we need some preparations. Let $Q$ be a probability measure on the space $(\Omega, \mathcal{F})$ such that $Q \ll P$. Then the following facts (a)-(f) are well-known (see Liptzer and Shiryaev 1974, for example).

(a)Set $q = \frac{dQ}{dP}$ and $q(t) = E[q|\mathcal{F}_t]$. Then $q(t)$ is a (not necessarily square integrable) $(\mathcal{F}_t, P)$ martingale.

(b) $q(t)$ has the following representation (Liptzer and Shiryaev p. 171 Theorem 5.8)

$$q(t) = 1 + \sum_{j=1}^{d} \int_0^t g_j(s)dW_j(s), \quad P(\int_0^T |g(s)|^2 ds = \int_0^T \sum_{j=1}^{d} |g_j(s)|^2 ds < \infty) = 1$$

(c)Set

$$\gamma_j(t) = \begin{cases} \frac{g_j(t)}{q(t)} & if \quad q(t) > 0 \\ 0 & otherwise \end{cases}$$

Then the process $\tilde{W}(t) = (\tilde{W}_1(t), \ldots, \tilde{W}_d(t))$ defined by

$$\tilde{W}_j(t) = W_j(t) - \int_0^t \gamma_j(s)ds$$

is a d-dimensional $(\mathcal{F}_t, Q)$-Wiener process (Liptzer and Shiryaev p. 225 Theorem 6.2).

(d)Using the notations of (c), we obtain the following formula

$$\begin{aligned} dS_k(t) &= b_k(t, S(t))dt + \textstyle\sum_{j=1}^d a_{kj}(t, S(t))dW_j(t) \\ &= \{b_k(t, S(t)) + \textstyle\sum_{j=1}^d a_{kj}(t, S(t))\gamma_j(t)\}dt + \textstyle\sum_{j=1}^d a_{kj}(t, S(t))\tilde{W}_j(t), \\ & \quad k = 1, \ldots, K. \end{aligned}$$

in the sense of probability measure Q. Therefore we know that, since we have assumed the Lipschitz continuity of $a_{kj}$, the necessary and sufficient condition for $Q$ to be a $S$-martingale measure is that

$$b_k(t, S(t)) + \sum_{j=1}^{d} a_{kj}(t, S(t))\gamma_j(t) = 0 \quad (dt \times dQ\, a.s.), \quad k = 1, \ldots, K.$$

From the definition of $\gamma_j(t), j = 1, \ldots, d$, the above condition is equivalent to

$$b_k(t, S(t))q(t) + \sum_{j=1}^{d} a_{kj}(t, S(t))g_j(t) = 0 \quad (dt \times dQ\, a.s.), \quad k = 1, \ldots, K.$$

In the case of $Q \sim P$, the following results hold true.

(e) If $Q \sim P$, then $q(t) > 0$ and $Q$ is obtained by Girsanov's transformation from $P$, namely, $q(t)$ has the following representation (Liptzer and Shiryaev p. 171 Theorem 5.9)

$$q(t) = \exp\{\int_0^t \sum_{j=1}^{d} \gamma_j(s)dW_j(s) - \frac{1}{2}\int_0^t \sum_{j=1}^{d}(\gamma_j(s))^2 ds\}$$

(f)If $Q \sim P$, then the necessary and sufficient condition for $Q$ to be a $S$-martingale measure is that

$$b_k(t, S(t)) + \sum_{j=1}^{d} a_{kj}(t, S(t))\gamma_j(t) = 0 \quad (dt \times dP\, a.s.), \quad k = 1, \ldots, K.$$

or equivalently

$$b_k(t, S(t))q(t) + \sum_{j=1}^{d} a_{kj}(t, S(t))g_j(t) = 0 \quad (dt \times dP\, a.s.), \quad k = 1, \ldots, K.$$

The following lemmma is essential for the proof of Theorem 4.

**Lemma 5** *Let $\{q_n(\omega), n = 1, 2, \ldots\}$ be a sequence in $L^1(\Omega, \mathcal{F}, P)$ such that $q_n(\omega) \geq 0$ and $\int_\Omega q_n(\omega)dP = 1$, and assume that*

$$q_n \to \hat{q} \text{ (as } n \to \infty) \qquad \text{in } L^1(\Omega, \mathcal{F}, P).$$

*Set $q_n(t) = E[q_n|\mathcal{F}_t]$ and $\hat{q}(t) = E[\hat{q}|\mathcal{F}_t]$, and using the fact (b), we represent $q_n(t)$ and $\hat{q}(t)$ in the following form*

$$q_n(t) = 1 + \sum_{j=1}^{d}\int_0^t g_j^{(n)}(s)dW_j(s), \quad P(\int_0^T |g^{(n)}(s)|^2 ds = \int_0^T \sum_{j=1}^{d} |g_j^{(n)}(s)|^2 ds < \infty) = 1$$

$$\hat{q}(t) = 1 + \sum_{j=1}^{d}\int_0^t \hat{g}_j(s)dW_j(s), \quad P(\int_0^T |\hat{g}(s)|^2 ds = \int_0^T \sum_{j=1}^{d} |\hat{g}_j(s)|^2 ds < \infty) = 1$$

*Then it holds that*

$$g_j^{(n)} \to \hat{g}_j \text{ (as } n \to \infty) \quad \text{in measure w.r.t. } dt \times dP, \quad j = 1, 2, \ldots, d.$$

(Proof) Let $\tau_l^{(n)}$ be the stopping time defined by

$$\tau_l^{(n)} = \begin{cases} \inf\{t \le T; q_n(t) \ge l\} \\ T, \quad if \quad \{t \le T;\ q_n(t) \ge l\} = \emptyset \end{cases}$$

and set $q_{n,l}(t) = q_n(t \wedge \tau_l^{(n)})$. Since $q_{n,l}(t)$ is square integrable, it has the following representation

$$q_{n,l}(t) = 1 + \sum_{j=1}^{d} \int_0^t g_j^{(n,l)}(s) dW_j(s)$$

and it holds that

$$g_j^{(n,l)} \to g_j^{(n)}\ (as \quad l \to \infty) \qquad in\ measure\ w.r.t.\ dt \times dP, \quad j = 1, \dots, d.$$

Moreover, as Liptser-Shiryaev showed (see Liptser-Shiryaev 1974, pp. 168-169), we can assume that

$$g_j^{(n)}(t, \omega) = g_j^{(n,l+k)}(t, \omega),\ k = 0, 1, \dots, \quad if \ \sup_{u \le t} |q_n(u, \omega)| \le l, \quad j = 1, \dots, d. \tag{3}$$

In the same way, setting

$$\hat{\tau}_l = \begin{cases} \inf\{t \le T;\ \hat{q}(t) \ge l\} \\ T, \quad if \quad \{t \le T;\ \hat{q}(t) \ge l\} = \emptyset \end{cases}$$

and $\hat{q}_l(t) = \hat{q}(t \wedge \hat{\tau}_l)$, we know that $\hat{q}_l$ is square integrable and that

$$\hat{q}_l(t) = 1 + \sum_{j=1}^{d} \int_0^t \hat{g}_j^{(l)}(s) dW_j(s)$$

and we can assume that

$$\hat{g}_j(t, \omega) = \hat{g}_j^{(l+k)}(t, \omega),\ k = 0, 1, \dots, \quad if \ \sup_{u \le t} |\hat{q}(u, \omega)| \le l, \quad j = 1, \dots, d.$$

We next prove that $g_j^{(n,l)}$ converges to $g_j^{(n)}$ in probability as $l \to \infty$ and that this convergence is uniform in $n, n = 1, 2, \dots,$. In fact, from the definition of $g_j^{(n,l)}$ and (3) we obtain for any $\epsilon > 0$

$$\begin{aligned} P(\sup_{u \le t} |g_j^{(n,l)}(u) - g_j^{(n)}(u)| > \epsilon) &\le P(\sup_{u \le t} q_n(u) > l) \\ &\le \tfrac{1}{l} \textstyle\int_{\sup\{q_n(u);\ u \le t\} > l} q_n(t) dP \\ &\le \tfrac{1}{l} E[q_n] \\ &\le \tfrac{1}{l} M, \qquad M = \sup_n E[q_n] \end{aligned} \tag{4}$$

where we use the martingale inequaliy. In the same way we know that for any $\epsilon > 0$ it holds that

$$P(\sup_{u \le t} |\hat{g}_j^{(l)}(u) - \hat{g}_j(u)| > \epsilon) \le \frac{1}{l} E[\hat{q}] \tag{5}$$

From (4) and (5) it follows that for any $\epsilon > 0$ and $\delta > 0$ there exists a consant $l_0 = l_0(\epsilon, \delta)$ such that for any $l \geq l_0$ it holds that

$$P(\sup_{u \leq t} |g_j^{(n,l)}(u) - g_j^{(n)}(u)| > \epsilon) < \delta, \quad n = 1, 2, \ldots, \quad j = 1, 2, \ldots, d, \tag{6}$$

and

$$P(\sup_{u \leq t} |\hat{g}_j^{(l)}(u) - \hat{g}_j(u)| > \epsilon) < \delta, \quad j = 1, 2, \ldots, d. \tag{7}$$

Next we will prove that, when $l$ is fixed, $g_j^{(n,l)}$ converges to $\hat{g}_j^{(l)}$ in probability as $n \to \infty$. From the definitions of $\{q_{n,l};\ n = 1, 2, \ldots\}$ and $\hat{q}_l$, they are square ntegrable and $q_{n,l} \to \hat{q}_l$ (as $n \to \infty$ ) in $L^2(dt \times dP)$. In the square integrable case, it is well-known that $g_j^{(n,l)} \to \hat{g}_j^{(l)}$ (as $n \to \infty$) in $L^2(dt \times dP)$ follows from $q_{n,l} \to \hat{q}_l$ (as $n \to \infty$) in $L^2(dt \times dP)$.Therefore we can result that for any fixed $l, l = 1, 2, \ldots$,

$$g_j^{(n,l)} \to \hat{g}_j(l) \quad (as\ n \to \infty)\ in\ measure\ w.r.t.\ dt \times dP, \quad j = 1, 2, \ldots, d. \tag{8}$$

Using the above results, we next prove that $g_j^{(n)} \to \hat{g}_j$ (as $n \to \infty$) in measure w.r.t. $dt \times dP$. We set $d\mu = dt \times dP$ on $[0, T] \times \Omega$. From (6) and (7) it folloes that for any $\epsilon > 0$ and $\delta > 0$,there exists a consant $l_0$ such that for any $l \geq l_0$ it holds that

$$\begin{gathered} \mu(\{(t, \omega) \in [0, T] \times \Omega;\ |g_j^{(n,l)}(t, \omega) - g_j^{(n)}(t, \omega)| > \epsilon\}) < T\delta, \\ n = 1, 2, \ldots, \quad j = 1, 2, \ldots, d, \end{gathered} \tag{9}$$

and

$$\mu(\{(t, \omega) \in [0, T] \times \Omega;\ |\hat{g}_j^{(l)}(t, \omega) - \hat{g}_j(t, \omega)| > \epsilon\}) < T\delta, \quad j = 1, 2, \ldots, d. \tag{10}$$

We here fix an $l, l \geq l_0$, and then using (8) we can take a number $n_0 = n_0(\epsilon, \delta, l)$ such that for any $n \geq n_0$

$$\mu(\{(t, \omega) \in [0, T] \times \Omega;\ |g_j^{(n,l)}(t, \omega) - \hat{g}_j^{(l)}(t, \omega)| > \epsilon\}) < \delta, \quad j = 1, 2, \ldots, d. \tag{11}$$

Using the results (9),(10) and (11), we obtain such a result that for any $\epsilon > 0$ and $\delta > 0$ we can choose constants $l_0 = l_0(\epsilon, \delta)$ and $n_0 = n_0(\epsilon, \delta, l), l \geq l_0$ such that for any $n \geq n_0$

$$\begin{aligned} \mu(|g_j^{(n)} - \hat{g}_j| > 3\epsilon) &\leq \mu(|g_j^{(n)} - g_j^{(n,l)}| > \epsilon) + \mu(|g_j^{(n,l)} - \hat{g}_j^{(l)}| > \epsilon) + \mu(|\hat{g}_j^{(l)} - \hat{g}_j| > \epsilon) \\ &\leq T\delta + \delta + T\delta = (2T + 1)\delta, \quad j = 1, 2, \ldots, d. \end{aligned}$$

Thus we have proved that $g_j^{(n)} \to \hat{g}_j$ in measure w.r.t. $dt \times dP$. (Q.E.D.)

**Remark 2** *Karatzas, Ocone and Li 1991 discussed similar problems. If we could prove that $q_n \in \mathbf{D}_{1,1}$ and that $q_n$ converges in $\mathbf{D}_{1,1}$ (with their notations), then we could apply their results. But it is not clear that $q_n \in \mathbf{D}_{1,1}$.*

(Proof of Theorem 4) Let $Q_n, 1, 2, \ldots$, be a sequence from $\mathcal{M}(S)$ such that $H(Q_n|P) \downarrow H(\mathcal{M}(S)|P)$. By Theorem 1, we can assume that $Q_n \in \mathcal{P}(S)$. Set $q_n(\omega) = \frac{dQ_n}{dP}$. Just as we have seen in the proof of Theorem 3, we know that there exists a probability measure $\hat{Q}$ such that $\hat{Q} \ll P$ and $Q_n \to \hat{Q}$ in variation, and

$$H(\hat{Q}|P) \leq \liminf_{n\to\infty} H(Q_n|P) = H(\mathcal{P}(S)|P) = H(\mathcal{M}(S)|P) \tag{12}$$

and that

$$\|Q_n - \hat{Q}\|_{var} = \int_\Omega |q_n - \hat{q}| dP(\omega) \to 0, \quad where \quad \hat{q} = \frac{d\hat{Q}}{dP}.$$

In the followings, we use the notations $(q_n, g_j^{(n)}, \gamma_j^{(n)}, \tilde{W}_j^{(n)})$ or $(\hat{q}, \hat{g}_j, \hat{\gamma}_j, \tilde{\hat{W}}_j)$, for $Q_n$ or $\hat{Q}$, corresponding to the notations $(q, g_j, \gamma_j, \tilde{W}_j)$ used in (a)-(f) for $Q$.

Since $Q_n \in \mathcal{P}(S)$, it follows from (f) that $q_n$ and $g_j^{(n)}$ satisfy the following equations

$$b_k(t, S(t))q_n(t) + \sum_{j=1}^d a_{kj}(t, S(t))g_j^{(n)}(t) = 0 \quad (P\,a.s.), \quad k = 1, \ldots, K. \tag{13}$$

By Lemma 5 it holds that

$$\lim_{n\to\infty} g_j^{(n)} = \hat{g}_j(in\ measure\ w.r.t.\ dt \times dP), \quad j = 1, \ldots, d \tag{14}$$

Therefore, taking a subsequence if necessary, from (13) and (14) we have obtained the following result

$$b_k(t, S(t))\hat{q}(t) + \sum_{j=1}^d a_{kj}(t, S(t))\hat{g}_j(t) = 0 \quad (dt \times dP\,a.s.), \quad k = 1, \ldots, K.$$

Since $\hat{Q} \ll P$, from this formula and (d) it follows that $\hat{Q}$ is $S$-martingale measure. Thus we have proved that $\hat{Q} \in \mathcal{M}(S)$, and that $\hat{Q}$ is the minimum point in $\mathcal{M}(S)$ by (12). The fact that $\hat{Q}$ is also the minimum point in $\mathcal{P}(S)$ follows from Theorem 2. The fact that $\hat{Q}$ is obtained by Girsanov's transformation from the original probability measure $P$, is follows from (e). The proof is complete. (Q.E.D.)

**Remark 3** *Föllmer and Schweizer 1990 discussed the similar problem in the space* $\mathcal{P}_2(S) = \{Q \in \mathcal{P}(S); S$ *is square integrable w.r.t.* $Q$ *and* $\frac{dQ}{dP} \in L^2(P)\}$ *which is a subspace of* $\mathcal{P}(S)$ *and* $\mathcal{M}(S)$. *They introduced the concept of minimal martingale measure(mMM) in* $\mathcal{P}_2(S)$, *and gave the sufficient conditions for the existence of mMM. We can verify that, when the system (1) is of simple form (for example* $b(t, S)$ *and* $a(t, S)$ *are linear in* $S$*), the mMM is identified with canonical martingale measure. So we come to the following conjecture.*

**Conjecture.** If mMM exists, then mMM is identified with the canonical maltingale measure.

## 4. Concluding Remarks

In this paper we have investigated the existence problem of canonical martingale measures. All our results are obtained without the assumption of the square integrability of Radon-Nikodym derivative $\frac{dQ}{dP}$. So our methods can be applied to many more cases. And if it is necessary to extend the class of martingale measures to the class of local martingale measures, then it may be possible to apply our methods to that of local martingale measures.

In the case of SDE discussed in section 3, if $d \leq K$, then the equivalent $S$-martingale measure is unique in general. Therefore, in order that the assets markets are incomplete, it must hold that $d > K$. On the other hand, if the price process $S(t)$ is a jump process, then the equivalent $S$-martingale measures are not unique (without the assumption of $d \leq K$) in general. So it may be an interesting problem to apply our methods to such processes. This is our next subject to study.

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### References

1. I. Csiszár, *Annals of Prob.* **3(1)** (1975), pp. 146-158.
2. I. Csiszár, *Annals of Prob.* **12(3)** (1984), pp. 768-793.
3. H. Föllmer and M. Schweizer, in *Applied Stochastic Analysis*, ed. M.H.A. Davis and R.J. Elliot ( Gordon and Breach, 1990), pp. 389-414.
4. J. M. Harrison and S. R. Pliska, *Stochastic Processes and Appl.* **11**(1981), pp. 215-260.
5. J. M. Harrison and S. R. Pliska, *Stochastic Processes and Appl.* **15**(1983), pp. 313-316.
6. S. Ihara, *Information Theory for Continuous Systems* (World Scientific, 1993), p. 308.
7. I. Karatzas, D. L. Ocone and J. Li, *Stochastics and Stochastics Reports* **37** (1991), pp. 127-131.
8. R. S. Liptser and A. N. Shiryaev, *Statistics of Random Processes: 1 General Theory* ( Springer-Verlag, 1974), P. 394.