Minimal Relative Entropy Martingale Measures of Geometric Lévy Processes and Option Pricing Models in Incomplete Markets

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Abstract

In this article the [Geometric Lévy Process & MEMM] pricing model is proposed. This model is an option pricing model for the incomplete markets, and this model is based on the assumptions that the price processes are geometric Lévy processes and that the prices of the options are determined by the minimal relative entropy methods.

This model has many good points. For example, the theoretical part of the model is contained in the framework of the theory of Lévy process (additive process). In fact the price process is also a Lévy process (with changed Lévy measure) under the minimal relative entropy martingale measure (MEMM), and so the calculation of the prices of options are reduced to the computation of functionals of Lévy process.

In a previous paper, we have investigated these models in the case of jump type geometric Lévy processes. In this paper we extend the previous results for more general type of geometric Lévy processes.

Key words: incomplete market, geometric Lévy process, relative entropy, martingale measure.

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1 Introduction

In the field of option pricing theory, the martingale measures play very important roles. For example, in the Black and Sholes model the price of a contingent claim is given as the expectation of the return function with respect to the unique risk neutral martingale measure.

If the market is complete, then the equivalent martingale measure is unique, and so the prices of options are uniquely determined by this martingale measure. However, if the market is incomplete, then there are (infinitely) many equivalent martingale measures. Therefore the pricing models for the incomplete markets are, in general, consisting of the following two parts. The first part is the part of defining the price process of underlying assets, and the second part is the part of selecting a suitable martingale measure, which determines the option prices, among the set of all equivalent martingale measures. And in order to accomplish the modeling problem, we have to prove the existence theorem of the specified martingale measure mentioned in the second part of the construction of the model (see [1], [13]).

Many kinds of processes are proposed as the examples of price processes of underlying assets. For example, diffusion processes, jump type processes and general semi-martingale processes are proposed. In this article we adopt the geometric Lévy processes as the underlying price processes.

For the problem to select a suitable martingale measure described in the second part of the construction of models, there are several candidates. For examples, minimal martingale measure, variance optimal martingale measure and utility martingale measure are proposed and discussed (see [3], [4], [5], etc.). In this article we adopt the minimal entropy martingale measure (MEMM) as the suitable martingale measure. The MEMM has been discussed in [8] and etc.

In the above context we have given two contents, a geometric Lévy process and the MEMM, and these two contents determine a pricing model of options. We name this model as [Geometric Lévy & MEMM] pricing model. The simplest case of this model is studied in [10] and [11]. Recently the similar problem has been investigated by T. Chan in [2].

In §2 we prepare the notations and definitions used in the following sections. In §3 we give the existence theorem of the MEMM and investigate the properties of the MEMM. The condition for the existence of MEMM of our theorem is almost the same as one of the results of Chan [2], but the methods of the proof is different.

In §4 we construct the [Geometric Lévy & MEMM] pricing model and explain the methods to apply this model to option pricing theory. Finally in §5 we give several remarks.
2 Preliminaries

Suppose that a probability space \((\Omega, \mathcal{F}, P)\) and an increasing family of sub \(\sigma\)-fields of \(\mathcal{F}\), \(\{\mathcal{F}_t, 0 \leq t \leq T\}\), are given as usual. In this paper we consider the case where the price process \(S(t)\) is in the form of \(S(t) = \exp[X(t)]\), where \(X(t)\) is a \(\mathcal{F}_t\) adapted Lévy process. It is well-known that a Lévy process \(X(t)\) is expressed in the following form (see e.g. [6])

\[
X(t) = X(0) + bt + \sigma W(t) + \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} x I_{\{|x| \geq 1\}} N_p(dsdx) \\
+ \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} x I_{\{|x| < 1\}} \tilde{N}_p(dsdx),
\]

where \(b\) and \(\sigma\) are constants, \(W(t)\) is a Wiener process, \(N_p(dsdx)\) is a counting measure of a Poisson point process, and \(\tilde{N}_p(dsdx)\) is the compensator of \(N_p(dsdx)\). In this case, the compensator \(\tilde{N}_p(dsdx)\) is in the form of \(\tilde{N}_p(dsdx) = N_p(dsdx) - \hat{N}_p(dsdx)\). The measure \(\nu(dx)\) is called Lévy measure of the process \(X(t)\), and it satisfies the following condition

\[
\int_{(-\infty,\infty) \setminus \{0\}} \{x^2/(1 + x^2)\} \nu(dx) < \infty.
\]

(2)

It is also assumed that

\[
\mathcal{F}_t = \sigma\{W(s), N_p(s, E) \mid s \leq t, E: Borel set\}, \quad 0 \leq t \leq T.
\]

(3)

We set the following assumption.

**Assumption 1** The Lévy measure \(\nu(dx)\) satisfies in addition to the above condition (2) the following condition

\[
\int_{0 < |x| < 1} |x| \nu(dx) < \infty.
\]

(4)

If the above assumption is satisfied, then \(\int_{(-\infty,\infty) \setminus \{0\}} x N_p(dsdx)\) is well-defined and the Lévy process \(X(t)\) is of the following form

\[
X(t) = X(0) + \gamma_0 t + \sigma W(t) + \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} x N_p(dsdx),
\]

where \(\gamma_0\) is a constant. The triplet \((\gamma_0, \sigma^2, \nu(dx))\) is called the characteristic triplet of \(X(t)\).

Next we give the definition of MEMM. We define \(\mathcal{P}(S)\) as the set of all equivalent \(S\)-martingale measures, namely the set of all probability measure \(Q\) on \((\Omega, \mathcal{F})\) such that \((S(t), 0 \leq t \leq T)\) is \((\mathcal{F}_t, Q)\)-martingale and \(Q \sim P\) (mutually absolutely continuous).
Definition 1 (minimal entropy martingale measure (MEMM)) If an equivalent martingale measure \( \hat{P} \) satisfies the following condition
\[
H(\hat{P}|P) \leq H(Q|P) \quad \forall Q \in \mathcal{P}(S) \tag{6}
\]
where \( H(Q|P) \) is the relative entropy of \( Q \) with respect to \( P \), which is given by the following formula
\[
H(Q|P) = \begin{cases} 
\int_{\Omega} \log \left( \frac{dQ}{dP} \right) dQ, & \text{if } Q \ll P, \\
\infty, & \text{otherwise},
\end{cases} \tag{7}
\]
then \( \hat{P} \) is called the minimal entropy martingale measure (MEMM) of \( S(t) \).

3 Main Results

In this section we investigate the existence theorem of MEMM of geometric Lévy processes under the setup of the previous section.

Let \( X(t) \) be a Lévy process which satisfies the Assumption 1. Then \( X(t) \) is expressed in the form of (5), namely
\[
X(t) = X(0) + \gamma_0 t + \sigma W(t) + \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} x N_p(dsdx). \tag{8}
\]

Our main result is the following theorem.

Theorem 1 Let \( S(t) = S_0 e^{X(t)} \) be a geometric Lévy process, where \( X(t) \) is a Lévy process given by (8).

Assume that there exist a non-positive constant \( \beta \) which satisfies the following equation
\[
\gamma_0 + \left( \frac{1}{2} + \beta \right) \sigma^2 + \int_{(-\infty,\infty) \setminus \{0\}} \left( e^x - 1 \right) e^{\beta(e^x-1)} \nu(dx) = 0, \tag{9}
\]
and let \( P^* \) be the probability measure defined by
\[
\frac{dP^*}{dP} \bigg|_{\mathcal{F}_t} = \exp \left[ \beta \sigma W(t) - \frac{1}{2} (\beta \sigma)^2 t + \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} \beta (e^x - 1) N_p(dsdx) - \right.
\]
\[
- \int_{[0,t]} \int_{(-\infty,\infty) \setminus \{0\}} (e^{\beta(e^x-1)} - 1) \nu(dx) \bigg], \tag{10}
\]
here we assume that the probability measure \( P^* \) is well-defined.

Then it holds that
(a)(MEMM): \( P^* \) is the MEMM of \( S(t) \).
(b) (Minimal Relative Entropy):

\[ H(P^*|P) = -T[\beta(\gamma_0 + (1 + \beta^2)\sigma^2) + \int_{(\gamma_0)}^\infty (e^{\beta(e^x-1)} - 1) \nu(dx)]. \]  

(11)

(c) (Lévy process): \(X(t)\) is also a Lévy process w.r.t. \(P^*\), and the characteristic triplet \((\gamma_0^*, a^*, \nu^*)\) of \(X(t)\) under \(P^*\) is

\[ \gamma_0^* = \gamma_0 + \beta\sigma^2, \quad a^* = a(= \sigma^2), \quad \nu^*(dx) = e^{\beta(e^x-1)} \nu(dx). \]  

(12)

**Remark 1**

(1) The result (a) of the Theorem is the same as one of the Chan’s results in [2]. His proof is based on the general results of semimartingales. We give more simple or elementary proof of it below.

(2) The result (c) is very important for the applications of our model to the option pricing theory as we see in the next section.

(3) The solution \(\beta\) of (9) is unique if it exists, because the left hand side of the equation (9) is an increasing function of \(\beta\) except for the case such that \(\sigma^2 = 0\) and \(\nu(dx) \equiv 0\).

Before we prove the theorem, we prepare the following lemmas.

**Lemma 1** (Fundamental Lemma, see [10] for example) Let \(P, Q,\) and \(\tilde{Q}\) be probability measures defined on a probability space \((\Omega, \mathcal{F})\) such that \(P \sim Q \sim \tilde{Q}\). Then it holds that

\[ H(Q|P) \geq \int_{\Omega} \log \left| \frac{d\tilde{Q}}{dP} \right| dQ. \]  

(13)

**Lemma 2** Let \(\mu(dx,dy)\) be a probability measure on \(R^2_+ = (0,\infty) \times (0,\infty)\) such that

\[ \int_0^\infty (y-x)d\mu_x(dy) = 0, \quad \tilde{\mu}(dx) - a.e., \]  

(14)

where \(\mu_x(dy)\) is the conditional probability and \(\tilde{\nu}(dx)\) is the marginal distribution of \(\nu\). Then \(\frac{(y-x)}{x}\) is integrable w.r.t. \(\mu(dx,dy)\), and it holds that

\[ \int_{R^2_+} \frac{(y-x)}{x} \mu(dx,dy) = 0. \]  

(15)

(Proof) We first see the integrability of \(f(x,y) = \frac{(y-x)}{x}\). Set

\[ f^{(+)}(x,y) = \max\{f(x,y),0\}, \quad f^{(-)}(x,y) = \max\{-f(x,y),0\}. \]

Since \(0 \leq f^{(-)}(x,y) \leq 1\), the integrability of \(f^{(-)}(x,y)\) is trivial. So we next see the integrability of \(f^{(+)}(x,y)\). Set

\[ f^{(+)}_K(x,y) = \min\{f^{(+)}(x,y), K\}, \quad 1 < K < \infty, \]
then, for the proof of the integrability of $f^+(x, y)$, we have only to prove that \( \{ \iint_{R^2_+} f^+_K(x, y) \mu(dxdy), K \in (1, \infty) \} \) is bounded.

For the fixed $K$, $1 < K < \infty$, we obtain

\[
\int \int_{R^2_+} f^+_K(x, y) \mu(dxdy) = \int \int_{0 < y-x < Kx} \frac{(y-x)}{x} \mu(dxdy) + \int \int_{y-x \geq Kx} K \mu(dxdy)
= \int_0^\infty \left\{ \int_{(K+1)x} x \frac{(y-x)}{x} \mu_x(dy) + \int_{(K+1)x}^\infty K \mu_x(dy) \right\} \bar{\mu}(dx)
= \int_0^\infty \left\{ \int_{0}^{(K+1)x} \frac{(y-x)}{x} \mu_x(dy) + \int_{(K+1)x}^\infty K \mu_x(dy) \right\} \bar{\mu}(dx)
+ \int \int_{R^2_+} f^-(x, y) \mu(dxdy),
\]

where we use the equality

\[
\int \int_{R^2_+} f^-(x, y) \mu(dxdy) = -\int_0^\infty \left\{ \int_0^x \frac{(y-x)}{x} \mu_x(dy) \right\} \bar{\mu}(dx).
\]

\( \xi \)From the assumption (14), we obtain

\[
\int_0^{(K+1)x} \frac{(y-x)}{x} \mu_x(dy) + \int_{(K+1)x}^\infty K \mu_x(dy) \leq \int_0^\infty \frac{(y-x)}{x} \mu_x(dy) = 0, \bar{\mu} - a.e.
\]

From (16) and (17) it follows that

\[
\int \int_{R^2_+} f^+_K(x, y) \mu(dxdy) \leq \int \int_{R^2_+} f^-(x, y) \mu(dxdy) \leq 1.
\]

This proves the integrability of the function $f(x, y)$. So, using the assumption (14) again, we obtain

\[
\int \int_{R^2_+} f(x, y) \mu(dxdy) = \int_0^\infty \left\{ \int_0^\infty f(x, y) \mu_x(dy) \right\} \bar{\mu}(dx)
= \int_0^\infty \left\{ \int_0^\infty \frac{(y-x)}{x} \mu_x(dy) \right\} \bar{\mu}(dx) = 0.
\]

The proof of Lemma 2 is completed. (Q.E.D.)
(Proof of Theorem 1)

Step 1. From Theorem 6.1 and Theorem 6.2 of Kunita-Watanabe[7], (or from Theorem 33.1 and Theorem 33.2 of K. Sato’s book[12]), we know that the Probability measure \( P^* \) is well defined and that under the new probability \( P^* X(t) \) is also a Lévy process with the triplet \((\gamma^*, a^*, \nu^*)\) in (c).

Step 2. From the results of Step 1, it follows that the generator of \( X(t) \) under \( P^* \) is

\[
L_X^{(P^*)} f(x) = \frac{1}{2} \sigma^2 \frac{d^2 f}{dx^2}(x) + (\gamma_0 + \beta \sigma^2) \frac{df}{dx}(x) + \int \{ f(x + y) - f(x) \} e^{\beta(e^y - 1)} \nu(dy). \tag{20}
\]

Therefore the generator of \( S(t) = S_0 e^{X(t)} \) under \( P^* \) is

\[
L_S^{(P^*)} F(S) = \frac{1}{2} \sigma^2 S^2 \frac{d^2 F}{dS^2}(S) + ((\gamma_0 + \beta \sigma^2) + \frac{1}{2} \sigma^2) S \frac{dF}{dS}(S) + \int \{ F(S e^y) - F(S) \} e^{\beta(e^y - 1)} \nu(dy) \tag{21}\]

In this formula, setting \( F(S) = S \) and using the formula (9), we know that \( L_S^{(P^*)} F(S) = 0 \). This proves that \( P^* \) is a martingale measure of \( S(t) \).

Step 3. Applying Itô’s formula to \( S(t) = S_0 e^{X(t)} \), we obtain

\[
dS(t) = S(t-)(\gamma_0 + \frac{1}{2} \sigma^2)dt + \sigma dW(t) + \int (e^x - 1) N_p(dtdx). \tag{22}\]

By this formula, the formula (10) is expressed in the following form

\[
\frac{dP^*}{dP} \bigg|_{\mathcal{F}_t} = \exp\left[ \int_{[0,t]} \frac{\beta}{S(s-)} dS(s) - \left\{ \beta(\gamma_0 + (\frac{1 + \beta}{2}) \sigma^2) + \int_{(-\infty,\infty)\setminus\{0\}} (e^{\beta(e^x - 1)} - 1) \nu(dx) \right\} t \right]. \tag{23}\]

We can now prove that \( P^* \) is the MEMM of \( S(t) \). Let \( Q \) be a \( S(t) \)-martingale measure such that \( Q \sim P \). Then, by the lemma stated above, we obtain

\[
H(Q|P) \geq \int_{\Omega} \log\left[ \frac{dP^*}{dP} \right] dQ. \tag{24}\]

We first consider such a case that \( \int_{[0,T]} \frac{\beta}{S(s-)} dS(s), 0 \leq t \leq T, \) is integrable with respect to \( Q \). In this case, by the assumption that \( Q \) is a martingale measure, it follows that

\[
E_Q\left[ \int_{[0,T]} \frac{\beta}{S(s-)} dS(s) \right] = 0. \tag{25}\]
Therefore we obtain from (22) that the right hand side of the inequality (24) is equal to

\[-T[\beta(\gamma_0 + (1+\beta/2)\sigma^2) + \int_{(-\infty,\infty)\setminus\{0\}} (e^{\beta(x^2-1)} - 1)\nu(dx)].\] (26)

On the other hand it is easy to see that the following equality holds

\[H(P^*|P) = -T[\beta(\gamma_0 + (1+\beta/2)\sigma^2) + \int_{(-\infty,\infty)\setminus\{0\}} (e^{\beta(x^2-1)} - 1)\nu(dx)].\] (27)

Thus we have proved that

\[H(Q|P) \geq H(P^*|P),\] (28)

for a martingale measure \(Q\) which satisfies the integrability assumption.

For the general martingale measures, we carry on the proof as follows by the use of discrete time approximation method. Suppose that a martingale measure \(Q\) is given and fixed, and let \(\Delta_n; n = 1, \ldots,\) be a series of partitions of the time interval \([0,T]\) such that

\[\Delta_n = \{t_i = iT/2^n, \ i = 0, 1, 2, \ldots, 2^n\}.\] (29)

Let \(\mathcal{F}(\Delta_n)\) be the natural sub \(\sigma\)-field of \(\mathcal{F}\) generated by the cylindrical subsets corresponding to \(\Delta_n\), and set

\[P_{\Delta_n} = P|_{\mathcal{F}(\Delta_n)}, \quad Q_{\Delta_n} = Q|_{\mathcal{F}(\Delta_n)}.\] (30)

Since \(Q \sim P\), it holds that \(Q_{\Delta_n} \sim P_{\Delta_n}\). From the well-known basic property of relative entropies, it follows that

\[H(Q_{\Delta_n}|P_{\Delta_n}) \uparrow H(Q|P) \quad (n \to \infty).\] (31)

Next we define \(P_n^*\) by

\[\frac{dP_n^*}{dP_{\Delta_n}|_{\mathcal{F}(\Delta_n)}} = \frac{\exp\{\beta\sum_{i=1}^{2^n}(e^{\Delta X_i} - 1)\}}{E_{P_{\Delta_n}}[\exp\{\beta\sum_{i=1}^{2^n}(e^{\Delta X_i} - 1)\}]} = \frac{\exp\{\beta\sum_{i=1}^{2^n}(e^{\Delta X_i} - 1)\}}{E_P[\exp\{\beta\sum_{i=1}^{2^n}(e^{\Delta X_i} - 1)\}]},\] (32)

where \(\Delta X_i = X(t_i) - X(t_{i-1})\). Since \(\beta \leq 0\), It is easy to see that the probability \(P_n^*\) on \(\mathcal{F}(\Delta_n)\) is well-defined by this formula, and that \(P_n^* \sim P_{\Delta_n}\). (Remark that \(P_n^*\) is an approximation of \(P^*\), but not necessarily \(P_n^* = P_{\Delta_n}^*\).) Applying Lemma 1 to \(P_{\Delta_n}, Q_{\Delta_n}\) and \(P_n^*\), we obtain

\[H(Q_{\Delta_n}|P_{\Delta_n}) \geq \int_{\Omega} \log\left|\frac{dP_n^*}{dP_{\Delta_n}}\right| dQ_{\Delta_n}.\] (33)
The right hand side of this inequality is equal to
\[ E_{Q_\Delta_n} \left[ \beta \sum_{i=1}^{2^n} (e^{\Delta X_i} - 1) \right] - \log \{ E_P[\exp \{ \beta \sum_{i=1}^{2^n} (e^{\Delta X_i} - 1) \}] \} \]
\[ = E_{Q_\Delta_n} \left[ \sum_{i=1}^{2^n} \frac{\beta}{S(t_{i-1})} (S(t_i) - S(t_{i-1})) \right] - \log \{ E_P[\exp \{ \beta \sum_{i=1}^{2^n} (e^{\Delta X_i} - 1) \}] \}. \quad (34) \]

Since \( Q \) is a martingale measure of \( S(t) \), \( Q_\Delta_n \) is a martingale measure of the discrete time process \( S(t_i) \). Therefore, using Lemma 2, we can prove easily that
\[ E_{Q_\Delta_n} \left[ \beta \sum_{i=1}^{2^n} \frac{\beta}{S(t_{i-1})} (S(t_i) - S(t_{i-1})) \right] = 0. \quad (35) \]

Thus we have obtained
\[ \int_{\Omega} \log \left[ \frac{dP^*_n}{dP_\Delta_n} \right] dQ_\Delta_n = - \log \{ E_P[\exp \{ \beta \sum_{i=1}^{2^n} (e^{\Delta X_i} - 1) \}] \}. \quad (36) \]

We will calculate the right hand side of the last equality. Using the properties of Lévy process, we obtain the following formula,
\[ - \log \{ E_P[\exp \{ \beta \sum_{i=1}^{2^n} (e^{\Delta X_i} - 1) \}] \} \]
\[ = - \log \left\{ \prod_{i=1}^{2^n} E_P[e^{\beta(e^{\Delta X_i} - 1)}] \right\} \]
\[ = -2^n \log \{ E_P[e^{\beta(e^{X(h)} - 1)}] \} = - \frac{T}{h} \log \{ E_P[e^{\beta(e^{X(h)} - 1)}] \}, \quad (37) \]

where \( h = \frac{T}{2^n} \). The generator of \( X(t) \) under \( P \) is
\[ L_X f(x) = \frac{1}{2} \sigma^2 \frac{d^2 f}{dx^2}(x) + \gamma_0 \frac{df}{dx}(x) + \int_{(\infty, \infty) \setminus \{0\}} (f(x + y) - f(x)) \nu(dy). \quad (38) \]

So applying this operator to \( f(x) = \exp \{ \beta(e^{x} - 1) \} \) and remembering that \( X(0) = 0 \), we obtain
\[ \lim_{h \downarrow 0} \frac{\log \{ E_P[e^{\beta(e^{X(h)} - 1)}] \}}{h} \]
\[ = \frac{1}{E_P[e^{\beta(e^{X(0)} - 1)}]} \lim_{h \downarrow 0} \frac{E_P[e^{\beta(e^{X(h)} - 1)}]}{h} \]
\[ = \beta(\gamma_0 + \frac{1 + \beta}{2} \sigma^2) + \int_{(\infty, \infty) \setminus \{0\}} (e^{\beta(y-1)} - 1) \nu(dy). \quad (39) \]
From (36), (37) and (39), we have obtained
\[
\lim_{n \to \infty} \int_{\Omega} \log \left[ \frac{dP_n^*}{dP_{\Delta n}} \right] dP_{\Delta n} = -T \{ \beta (\gamma_0 + \frac{1 + \beta}{2} \sigma^2) + \int_{(-\infty, \infty) \setminus \{0\}} (e^{\beta y} - 1) \nu(dy) \}. 
\]
(40)

From (31), (33) and (40), it follows that
\[
H(Q|P) \geq -T \{ \beta (\gamma_0 + \frac{1 + \beta}{2} \sigma^2) + \int_{(-\infty, \infty) \setminus \{0\}} (e^{\beta y} - 1) \nu(dy) \}. 
\]
(41)

This inequality and the formula (27) prove that \( P^* \) is the MEMM, and the proof of the theorem is completed. (Q.E.D.)

**Remark 2** In general, there are many martingale measures of \( S(t) = S_0 e^{X(t)} \).

Let \( \beta_1 \) and \( \beta_2 \) satisfy the following equation
\[
\gamma_0 + \left( \frac{1}{2} + \beta_1 \right) \sigma^2 + \int_{(-\infty, \infty) \setminus \{0\}} (e^x - 1) e^{\beta_2 (e^x - 1)} \nu(dx) = 0. 
\]
(42)

Then the probability measure \( Q(\beta_1, \beta_2) \) defined by
\[
\left. \frac{dQ(\beta_1, \beta_2)}{dP} \right|_{x_t} = \exp[\beta_1 \sigma W(t) - \frac{1}{2} (\beta_1 \sigma)^2 t + \int_{[0,t]} \int_{(-\infty, \infty) \setminus \{0\}} \beta_2 (e^x - 1) N_p(dsdx) - 
\int_{[0,t]} \int_{(-\infty, \infty) \setminus \{0\}} (e^{\beta_2 (e^x - 1)} - 1) \nu(dx) ] ,
\]
(43)
is a martingale measure of \( S(t) \). This fact can be proved in the same manner as Step 1 and Step 2 of the proof of Theorem 1.

4 Application to the Option Pricing Theory in Incomplete Markets

From Remark 1 we know that the market corresponding to geometric Lévy process \( S(t) \) is incomplete. As we have stated in §1, a pricing model of the incomplete market consists of two parts: (A) the price process of the underlying assets, and (B) the martingale measure which determines the prices of options.

Based on the above results (Theorem 1 and Remark 2), we adopt (A) the geometric Lévy process \( S(t) = S_0 e^{X(t)} \) as the price process of the underlying assets, and (B) the MEMM \( P^* \) as the martingale measure which determines the prices of options, and we name this model as the [Geometric Lévy & MEMM] pricing model.
Under the framework of the [Geometric Lévy & MEMM] pricing model, the price of an option $Z$ is given by the expectation $E_{P^*}[Z]$. In general an option is a functional of the price process $S(t)$. By Theorem 1, the process $S(t)$ is geometric Lévy process under the MEMM $P^*$. Therefore the calculation of option prices are reduced to the computation of the expectations of functionals of Lévy process. If an option is dependent on only the values $S(T)$, then the option price can be easily computed by the use of the distribution of $X(T)$. For example, the European call option depends on only the values $S(T)$, so the price of it is easily computed.

**Example 1 (Compound Poisson Process)** Suppose that the Lévy process $X(t)$ is compound Poisson process, and assume that the Lévy measure $\nu(dx)$ is given in the form of

$$\nu(dx) = c\sigma(dx), \quad (44)$$

where $\sigma(dx)$ is a probability measure on $(-\infty, \infty)$ such that $\sigma(\{0\}) = 0$, and $c$ is a positive constant. Then the corresponding compound Poisson process $X(t)$ is

$$X(t) = X(0) + \int_{[0,t]} \int_{(-\infty,\infty)\setminus\{0\}} xN_p(dsdx). \quad (45)$$

The equation (9) for $\beta$ in Theorem 1 is

$$\int_{(-\infty,\infty)\setminus\{0\}} (e^x - 1)e^{\beta(e^x-1)}\sigma(dx) = 0. \quad (46)$$

Suppose that this equation has a solution $\beta$, then by Theorem 1 (a) the MEMM, $P^*$, exists and by Theorem 1 (c) the process $X(t) = \log[S(t)/S_0]$ is also a compound Poisson process with Lévy measure $\nu^*(dx) = e^{\beta(e^x-1)}\nu(dx)$. Therefore, under the MEMM $P^*$, the process $X(t) = \log[S(t)/S_0]$ is a compound Poisson process with Lévy measure

$$\nu^*(dx) = e^{\beta(e^x-1)}\nu(dx). \quad (47)$$

In the above example, set $c = 1$, $\sigma(\{1\}) = \sigma(\{-1\}) = \frac{1}{2}$, namely $\nu(dx) = \frac{1}{2}\delta_1(dx) + \frac{1}{2}\delta_{-1}(dx)$. Then we obtain $\beta = \frac{-e}{(e+1)(e-1)}$, and

$$\nu^*(dx) = e^{\beta(e^x-1)}\nu(dx) = \frac{1}{2}\exp\left[\frac{-e}{e+1}\right]\delta_1(dx) + \frac{1}{2}\exp\left[\frac{1}{e+1}\right]\delta_{-1}(dx). \quad (48)$$
5 Concluding Remarks: Problems related to our Model

In order to apply our [Geometric Lévy & MEMM] pricing model to the pricing problem of contingent claims, we have to verify that this model fits to the real financial markets. To do this, we need to carry on the following investigations.

(1) The distribution of $X(t) = \log S(t)$ is supposed to be an infinitely divisible distribution. So we have to check that this assumption is true for many cases. And next we have to estimate the characteristic triplet of the infinitely divisible distributions.

(2) In order to calculate option prices, we have to compute the expectations of functionals of Lévy process. These computations can be done after the theory of stochastic analysis based on Lévy processes has been established. So we have to study the stochastic analysis theory based on Lévy process.

(3) Finally we have to compare the theoretical prices of options (derived by our model) with the empirical prices in the markets.

These three are the problems which we have to investigate in the future.

References


