Minimal Relative Entropy Martingale Measures and their Applications to Option Pricing Theory

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Abstract

In this paper we review the relative entropy meathods for the option pricing theory in the incomplete markets. First we summarize the known results with respect to the existence of minimal relative entropy martingale measure (MEMM), and then we give several examples of the pricing models related to the MEMM (for example, the [Geometric Lévy & MEMM] pricing model). After that we explain the mathematical and (or) financial problems relating to our models.

Key words: incomplete market, relative entropy, minimal entropy martingale measure (MEMM), geometric Lévy process.

1 Introduction

In the field of option pricing theory, the martingale measures play very important roles. For example, in the Black-Sholes model the price of a contingent claim is given as the expectation of the return function with respect to the unique risk neutral martingale measure.

If the market is complete, then the equivalent martingale measure is unique and the prices of options are uniquely determined by this martingale measure (as the case of Black-Sholes model). However, if the market is incomplete, then there are (infinitely) many equivalent martingale measures.

Therefore we have to select a suitable martingale measure, which determines the option prices, among the set of all equivalent martingale measures.

There are several candidates for the suitable martingale measure. For example, minimal martingale measure, variance optimal martingale measure and utility martingale measure are proposed and discussed (see [2], [5], [6], etc.). In this article we adopt the minimal entropy martingale measure (MEMM) as the suitable martingale measure. The MEMM has been discussed in [13] (where MEMM is called CMM) and etc. This measure has some relations with the Esscher Transforms (see [1]).

When the MEMM is adopted as the suitable martingale measure, we say that we use the relative entropy method. The pricing models for the incomplete markets are, in general, consisting of the following two parts. The first part is the part of defining the price process of underlying assets, and the second part is the part of selection of a suitable martingale measure (see [1], [20]). And in order to accomplish the modeling problem, we have to prove the existence of the specified martingale measure of the second part of the model.

As examples of price processes of underlying assets, many processes have been proposed. For example, diffusion processes, jump type processes and general semi-martingale processes are proposed. For some of them the existence of the MEMM has been proved. The most important model is the [Geometric Lévy & MEMM] pricing model, which was introduced by [16]. In this article we adopt the geometric Lévy processes as the underlying price processes.

In §2 we give the definition of the MEMM which shall be used in the following sections. In §3 we summarize the relative entropy methods in the pricing theory. In §4 we explain the [Geometric Lévy & MEMM] pricing model. InS5 finally we state several problems relating to our model.

2 The MEMM

Suppose that a probability space $(\Omega, \mathcal{F}, P)$ and an increasing family of sub σ-fields of $\mathcal{F}$, $\{\mathcal{F}_t, 0 \leq t \leq T\}$, are given as usual. A price process is a $\mathcal{F}_t$-adapted stochastic process $S(t)$ defined on $(\Omega, \mathcal{F}, P)$.

We define $\mathcal{P}(S)$ as the set of all equivalent S-martingale measures, namely the set of all probability measures Q on $(\Omega, \mathcal{F})$ such that $(S(t), 0 \leq t \leq T)$ is $(\mathcal{F}_t, Q)$-martingale and $Q \sim P$ (mutually absolutely continuous). If the set $\mathcal{P}(S)$ is singleton, then the market is said to be *complete*. If the set $\mathcal{P}(S)$ consist of more than one element, then the market is said to be *incomplete*.

Now we give the definition of the MEMM.

Definition 1 (minimal entropy martingale measure (MEMM)) *If an equivalent martingale measure $\hat{P}$ satisfies the following condition*

$$H(\hat{P}|P) \leq H(Q|P) \qquad \forall Q \in \mathcal{P}(S) \tag{1}$$

where $H(Q|P)$ is the relative entropy of Q with respect to P, which is given by the following formula

$$H(Q|P) = \begin{cases} \int_\Omega \log[\frac{dQ}{dP}]dQ, & if \quad Q \ll P, \\ \infty, & otherwise, \end{cases} \tag{2}$$

then $\hat{P}$ is called the minimal entropy martingale measure (MEMM) of $S(t)$.

The basic properties of MEMM are described in §2 of [13]. For example, it is known that if the MEMM exists then it is unique.

There are many reasons why we adopt the MEMM as the suitable equivalent martingale measure. One of them is the fact that the MEMM is related to the utility function of exponential type. And another one is the relations of the MEMM with the theory of large deviation (remember Sanov's Lemma).

3 Models of Incomplete Markets and Existence Theorems of MEMM

As we have stated in Section 1, a pricing model of the incomplete market consists of the following two parts,

(A) the price process of the underlying assets,

(B) the suitable martingale measure which determines the prices of options.

Since we have adopted the MEMM as the suitable martingale measure, we only have to give the price process $S(t)$ in order to determine a pricing model.

In this section we will give several examples of price processes and the existence theorems of the MEMM with respect to each of the price processes.

3.1 Diffusion model

Suppose that the price process $S(t)$ is given in the diffusion type stochastic differential equations.

Theorem 1 ([13, Theorem 4]) *Let $W(t) = (W_1(t), \ldots, W_{d_1}(t))$ be the d_1-dimensional $(\mathcal{F}_t, P)$ Wiener process, and assume that $\mathcal{F}_t = \mathcal{F}_t^W = \sigma\{W(s), s \leq t\}$. Suppose that the price process $S(t) = (S_1(t), \ldots, S_d(t))$ is given by*

$$S_i(t) = S_i(0) + \int_0^t b_i(s, S(s))ds + \sum_{j=1}^{d_1} \int_0^t a_{ij}(s, S(s))dW_j(s), i = 1, \ldots, d, \quad (3)$$

where we assume that b_i and $a_{ij}, i = 1, 2, \ldots, d, j = 1, 2, \ldots, d_1$, satisfy the global Lipschitz condition. If there exists a martingale measure $Q \in \mathcal{M}(S)$ such that $H(Q|P) < \infty$, then there exists the MEMM P^*, and P^* is obtained by Girsanov transformation from P.

Remark 1 *(a) In the above theorem, the square integrability of the density $\frac{dP^*}{dP}$ is not assumed.*
(b) If the system is incomplete (for example, the case of $d < d_1$), then Girsanov transformation is not necessarily unique. So the result of the above theorem insists that the MEMM is one of the measures which are obtained by Girsanov transformation from P.

3.2 Jump Markov Model

If the price process is a jump type process, then the model is incomplete in general. Many researchers have been investigated such cases for the study of incomplete markets. (See [11], etc.)

The existence theorems of MEMM for these processes have not yet obtained in a satisfied form and are open problems. Two cases are known. One is the case where the underlying price process is jump type geometric Lévy process (see Section 4). The other one is the case where the underlying price process is Birth and Death process. (See [17]).

For the discrete time Markov processes, Mister M. Matano's results are known. (See Theorem 2 and Acknowledgements in [15]).

3.3 Geometric Lévy Model

The case where the geometric Lévy process $S(t) = S_0 e^{X(t)}$ is adopted as the price process of the underlying assets, is discussed in [15] and [16].

This model is very important, because the distribution of Lévy process $X(t)$ is infinitely divisible distributed. The class of infinitely divisible distributions contains normal distributions, compound Poisson distributions, stable distributions and etc. Many researchers investigated the empirical distributions of the prices of financial assets, and many authors insisted that the empirical distributions are contained in the class of infinitely divisible distributions. (See [3], [4], [7], [10], [12], etc.)

We will discuss this model more precisely in the next section (see §4).

3.4 Other Models

Random volatility models are often discussed. But the MEMM has not yet been discussed for those models.

4 [Geometric Lévy & MEMM] pricing model

When we adopt a model such that

(A) the geometric Lévy process $S(t) = S_0 e^{X(t)}$, where $X(t)$ is Lévy process, as the price process of the underlying assets, and

(B) the MEMM P^* as the martingale measure which determines the prices of options,

we call this model as the [Geometric Lévy & MEMM] pricing model.

This model is investigated in [16], and the following theorem is obtained.

Theorem 2 ([16, Theorem 1]) *Let $S(t) = S_0 e^{X(t)}$ be a geometric Lévy process, where $X(t)$ is a Lévy process.*

Assume that there exist a constant β which satisfies the following equation

$$\gamma_0 + (\frac{1}{2} + \beta)\sigma^2 + \int_{(-\infty,\infty)\setminus\{0\}} (e^x - 1)e^{\beta(e^x-1)}\nu(dx) = 0, \tag{4}$$

and let P^* be the probability measure defined by

$$\left.\frac{dP^*}{dP}\right|_{\mathcal{F}_t} = \exp[\beta\sigma W(t) - \frac{1}{2}(\beta\sigma)^2 t + \int_{[0,t]}\int_{(-\infty,\infty)\setminus\{0\}} \beta(e^x - 1)N_p(dsdx) - \\ - \int_{[0,t]}\int_{(-\infty,\infty)\setminus\{0\}} (e^{\beta(e^x-1)} - 1)\nu(dx)], \tag{5}$$

here we assume that the probability measure P^* is well-defined.

Then it holds that

(a)(MEMM): P^* is the MEMM of $S(t)$.

(b)(Minimal Relative Entropy):

$$H(P^*|P) = -T[\beta(\gamma_0 + (\frac{1+\beta}{2})\sigma^2) + \int_{(-\infty,\infty)\setminus\{0\}} (e^{\beta(e^x-1)} - 1)\nu(dx)]. \tag{6}$$

(c) (Lévy process): $X(t)$ is also a Lévy process w.r.t. P^* and the triplet (γ_0^*, a^*, ν^*) is

$$\gamma_0^* = \gamma_0 + \beta\sigma^2, \quad a^* = a(= \sigma^2), \quad \nu^*(dx) = e^{\beta(e^x-1)}\nu(dx).$$

Under the framework of the [Geometric Lévy & MEMM] pricing model, the price of an option Z is given by the expectation $E_{P^*}[Z]$. In general an option is a functional of the price process $S(t)$. By Theorem 2, the process $S(t)$ is geometric Lévy process under the MEMM P^*. Therefore the calculation of option prices are reduced to the computation of the expectations of functionals of Lévy process.

Example 1 (Compound Poisson Process) *Suppose that the Lévy process $X(t)$ is compound Poisson process and that the Lévy measure $\nu(dx)$ is given in the form of*

$$\nu(dx) = c\sigma(dx), \tag{7}$$

where $\sigma(dx)$ is a probability measure on R such that $\sigma(\{0\}) = 0$, and c is a positive constant. Then the equation (10) for β in Theorem 1 is

$$\int_{(-\infty,\infty)\setminus\{0\}} (e^x - 1)e^{\beta(e^x-1)}\sigma(dx) = 0. \tag{8}$$

Suppose that this equation has a solution β, then by Theorem 1 (a) the MEMM, P^*, exists and by Theorem 1 (c) the process $X(t) = \log[S(t)/S_0]$ is also a compound Poisson process with Lévy measure $\nu^*(dx) = e^{\beta(e^x-1)}\nu(dx)$.

In the above example, set $c = 1$, $\sigma(\{1\}) = \sigma(\{-1\}) = \frac{1}{2}$, namely $\nu(dx) = \frac{1}{2}\delta_1(dx) + \frac{1}{2}\delta_{-1}(dx)$. Then we obtain $\beta = \frac{-e}{(e+1)(e-1)}$.

Therefore the process $X(t) = \log[S(t)/S_0]$ is a compound Poisson process with Lévy measure

$$\nu^*(dx) = e^{\beta(e^x-1)}\nu(dx) = \frac{1}{2}\exp[\frac{-e}{e+1}]\delta_1(dx) + \frac{1}{2}\exp[\frac{1}{e+1}]\delta_{-1}(dx). \tag{9}$$

¿From this we know that the distribution of $X(t)$ is the convolution of two mutually-independent Poisson distributions. Therefore the prices of options which are dependent on only the values $S(T)$ can be easily computed by the use of the distribution of $X(T)$. For example, the European call option depends on only the values $S(T)$, so the price of it is easily computed.

5 Concluding Remarks

As we have explained in the previous section, the [Geometric Lévy & MEMM] pricing model has many good points. Even though, we have to verify that this model fits to the real financial markets. In order to do this, we need to carry on the following investigations.

(1)The distribution of $X(t) = \log S(t)$ is supposed to be an infinitely divisible distribution. Therefore we have to check this fact and next have to estimate the characteristic triplet of the infinitely divisible distributions.

(2)In order to calculate option prices, we have to compute the expectations of functionals of Lévy process. This is the study of stochastic analysis based on Lévy processes.

(3) Finally we have to compare the theoretical prices of options with the market prices of them.

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