MINIMAL F^Q-MARTINGALE MEASURES FOR EXPONENTIAL LÉVY PROCESSES

By Monique Jeanblanc,
Susanne Klöppel* and Yoshio Miyahara†

Université d'Evry Val d'Essonne and Europlace de France, Vienna University of Technology, Nagoya City University

Let L be a multidimensional Lévy process under P in its own filtration. The f^q-minimal martingale measure Q_q is defined as that equivalent local martingale measure for $\mathcal{E}(L)$ which minimizes the f^q-divergence $E\big[(dQ/dP)^q\big]$ for fixed $q \in (-\infty, 0) \cup (1, \infty)$. We give necessary and sufficient conditions for the existence of Q_q and an explicit formula for its density. For $q = 2$, we relate the sufficient conditions to the structure condition. Moreover, we show that Q_q converges for $q \searrow 1$ in entropy to the minimal entropy martingale measure.

1. Introduction. Lévy models are very popular in finance due to their tractability and their good fitting properties. However, Lévy models typically yield incomplete markets. This raises the question of which measure one should choose for valuation or pricing of untraded payoffs. Very often, a measure is chosen which minimizes a particular functional over the set $\mathcal{M}^e(S)$ of equivalent local martingale measures for the underlying assets S. This choice can be motivated by a dual formulation of a primal utility maximization problem; see [14] and [6]. If P denotes the subjective measure, then the functional on $\mathcal{M}^e(S)$ is typically of the form $Q \mapsto E_P\big[f(dQ/dP)\big]$ where f is a convex function on $(0, \infty)$. Then $f(Q|P) := E_P\big[f(dQ/dP)\big]$, known as the *$f$-divergence* of Q with respect to P, is a measure for the distance between Q and P; see [17] for a textbook account. Hence one chooses as pricing measure the martingale measure which is closest to P with respect to some f-divergence.

In this article, we consider $f^q(Q|P)$ corresponding to $f^q(z) = z^q$ for $q \in (-\infty, 0) \cup (1, \infty)$. The corresponding optimal measure Q_q is called f^q-

*Financial support by the NCCR FINRISK and the Christian Doppler Research Association is gratefully acknowledged.

†Supported by Grant-in-Aid for Scientific Research No.16540113, JSPS.

AMS 2000 subject classifications: Primary 60G51; secondary 91B28

Keywords and phrases: Lévy processes, martingale measures, f^q-minimal martingale measure, variance minimal martingale measure, f-divergence, structure condition, incomplete markets

minimal martingale measure. More precisely, we work on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration which is the P-augmentation of that generated by a d-dimensional Lévy process L and model the traded assets S as the stochastic exponential $S = \mathcal{E}(L)$. Based on an explicit formula for $f^q(Q|P)$ in terms of the Girsanov parameters of Q, we show that $f^q(Q|P)$ is reduced if Q is replaced by some $\overline{Q} \in \mathcal{M}^e(S)$ which is defined via its Girsanov parameters $(\overline{\beta}, \overline{Y})$. More precisely, $(\overline{\beta}, \overline{Y})$ is obtained by a pointwise minimization of the expression for $f^q(Q|P)$; see Theorem 2.6. It turns out that Q_q preserves the Lévy property of L, i.e., that L is a Q_q-Lévy process, and that for minimization of $f^q(Q|P)$ it suffices to consider those $Q \in \mathcal{M}^e(S)$ which preserve the Lévy property of L. As illustrated in Theorem 2.7, this allows to reduce the minimization of $f^q(Q|P)$ to a deterministic convex optimization problem $(\mathcal{P}_q)$ whose solution corresponds to the Girsanov parameters of Q_q and hence provides an explicit formula for Q_q . In particular, $(\mathcal{P}_q)$ has a solution if and only if Q_q exists. By formally applying Kuhn-Tucker's theorem to $(\mathcal{P}_q)$ we obtain in Theorem 2.9 conditions $(\mathcal{C}_q)$, which are sufficient for the existence of a solution to $(\mathcal{P}_q)$ and can easily be verified in practice. From $(\mathcal{C}_q)$ one can immediately deduce the solution to $(\mathcal{P}_q)$, and hence obtains the explicit formula for Q_q. For $q = 2$, we quote in Theorem 3.1 several cases in which the sufficient conditions $(\mathcal{C}_2)$ are also necessary for the existence of Q_2, if an additional integrability condition on L holds; the latter condition is actually implied by $(\mathcal{C}_2)$ if $d = 1$. In addition, we relate Q_2 to the variance optimal signed martingale measure which in our setting exists, if the *structure condition* (SC) is satisfied. We show that (SC) is equivalent to $(\mathcal{C}_2)$ and the above mentioned integrability assumption on L. Finally, we prove that under some technical assumptions, Q_q converges for $q \searrow 1$ in entropy to the *minimal entropy martingale measure* P_e and that also the corresponding Girsanov parameters converge; P_e is defined as that measure Q which minimizes the divergence corresponding to $f(z) := z \log z$ over all local martingale measures for S. The convergence is shown by an application of the implicit function theorem; for this $(\mathcal{C}_q)$ has to be rewritten as the root of an appropriate function. A concluding example illustrates that the conditions we have to impose for the convergence can all be satisfied in a reasonable model.

Some of the results and concepts have been studied before. Esche and Schweizer use in [5] an approach similar to our Theorem 2.6 for P_e instead of Q_q. However, instead of defining the Girsanov parameters of $\overline{Q}$ via pointwise minimization, they average the Girsanov parameters of the original measure Q with respect to $dQ \otimes dP$. Therefore, they require an extensive approximation approach in order to prove that $\overline{Q}$ is in fact a martingale measure. We thus do not only generalize their approach to Q_q but, more importantly, significantly simplify it. Although the problem $(\mathcal{P}_q)$ itself seems not be studied before, the sufficient conditions $(\mathcal{C}_q)$ appear for $q \in (-\infty, 0)$ also in Kallsen [13] and Goll and Rüschendorf [9]. However, they do only state $(\mathcal{C}_q)$ without motivating its definition, which is done in our approach. Very recently, we got to know an article of Choulli, Stricker and Li [3] in which they obtain (C_q) for general q, but for the minimal Hellinger martingale measure of order q instead of Q_q, for discrete time market models they also prove that these measures are the same. Convergence of Q_q to P_e for $q \searrow 1$ was studied for continuous processes by Grandits and Rheinländer in [10], but an extension to processes which also have a jump part was still missing. The relation between $(\mathcal{C}_2)$ and the structure condition as well as a study of the conditions when $(\mathcal{C}_2)$ is even necessary for the existence of Q_2 seem to be new. Finally, we think it is remarkable that our approach is very intuitive and accessible to anybody interested in risk neutral measures.

The paper is structured as follows. In Section 2 we give necessary and sufficient conditions for the existence of Q_q and an explicit formula for its density. Section 3 covers additional results for the special case $q = 2$. Convergence of Q_q to P_e is presented in Section 4. Appendix A contains the required results on Lévy processes, Appendix B some auxiliary calculations, and Appendix C some proofs omitted from the main body the article.

2. Structure and existence of the f^q-minimal martingale measure. In this section we give necessary and sufficient conditions for the existence of the f^q-minimal martingale measure (qMMM) Q_q for some exponential Lévy process S; q is assumed to be in $I := (-\infty, 0) \cup (1, \infty)$. Moreover, we provide explicit formulas for the density of Q_q.

We start with some notation and conventions. Throughout the article we work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F} = \mathbb{F}^L$ is the P-augmentation of the filtration generated by a d-dimensional Lévy process $L = (L_t)_{0 \le t \le T}$ with characteristic triplet (b, c, K) with respect to the truncation function $h(x) := x\mathbf{I}_{\{\|x\| \le 1\}}$ and where T is a finite time horizon. The random measure associated with the jumps of L is denoted by μ^L and $\nu^P(dx, dt) = K(dx)dt$ is the predictable P-compensator of μ^L; all required background on Lévy processes and unexplained terminology can be found in Appendix A. We write $E_Q[\,.\,]$ for the expectation with respect to Q and $E[\,.\,]$ if $Q = P$. For any $Q \ll P$, its real-valued density process $Z^Q = (Z_t^Q)_{0 \le t \le T}$ with $Z_t^Q := E[dQ/dP|\mathcal{F}_t]$ is defined with respect to P. If not mentioned otherwise, processes are assumed to be $\mathbb{R}^d$-valued and if it exists, we choose a right-continuous version.

2.1. *Reducing the problem.* Our aim is to characterize for $q \in I$ the qMMM $Q_q \approx P$ which is for $f^q(Q|P) := E\big[f^q(Z_T^Q)\big]$ and $f^q(z) := z^q$ defined by the condition

$$f^q(Q_q|P) = \inf_{Q \in \mathcal{M}^e(S)} f^q(Q|P), \tag{2.1}$$

where $\mathcal{M}^e(S)$ denotes the set of all *equivalent* local martingale measures for the exponential Lévy process $S := \mathcal{E}(L) = \left(\mathcal{E}(L^1), \ldots, \mathcal{E}(L^d)\right)^*$; note that $\mathcal{M}^e(S) = \mathcal{M}^e(L)$. The process S is assumed to be strictly positive, i.e., $\Delta L > -1$ P-a.s. In (2.1) it is clearly sufficient to consider those Q with $f^q(Q|P) < \infty$, i.e., the set

$$\mathcal{Q}^q := \left\{ Q \in \mathcal{M}^e(S) \,|\, f^q(Q|P) < \infty \right\},$$

for which we make the standing

Assumption: $\mathcal{Q}^q \neq \emptyset.$

In our Lévy setting any $Q \approx P$ can be fully described by its Girsanov parameters (β, Y) with respect to L and we write $Q = Q^{(\beta,Y)}$ to emphasize this; see Proposition A.3. We show that in (2.1) it is even sufficient to consider $\mathcal{Q}^q \cap \overline{\mathcal{Q}}$ where

$$\overline{\mathcal{Q}} := \left\{ Q^{(\beta,Y)} \in \mathcal{M}^e(S) \,\middle|\, (\beta, Y) \text{ time-independent and deterministic} \right\},$$

is the set of all $Q \in \mathcal{M}^e(S)$ for which the P-Lévy process L is also a Q-Lévy process; see Corollary A.7 below. In particular, this implies that Q_q preserves the Lévy property of the P-Lévy process L, i.e.,, that L is also a Q_q-Lévy process. In addition, we prove that Q_q can even be fully described by the solution of a deterministic optimization problem in $\mathbb{R}^d$. In particular, Q_q exists if and only if this optimization problem has a solution.

Our first result yields an explicit formula for $f^q(Q|P) = E\big[f^q(Z_T^Q)\big] = E\big[(Z_T^Q)^q\big]$ with $q \in I$. In the whole section we assume that $q \in I$ is arbitrary but fixed. Moreover, we need the following function

$$g_q(y) := y^q - 1 - q(y-1).$$

Proposition 2.1. *Let $Q = Q^{(\beta,Y)} \in \mathcal{Q}^q$ with density process $Z = Z^Q = \mathcal{E}(N)$. The canonical P-decomposition of $f^q(Z) = Z^q = M + A$ is*

$$\begin{array}{rrcl} & M & = & \int Z^q_- \, d\widehat{M} = \int f^q(Z_-)\, d\widehat{M} \\ \textit{with} & \widehat{M} & := & q\,N + g_q(Y) * (\mu^L - \nu^P) = q\,N^c + (Y^q - 1) * (\mu^L - \nu^P) \\ \textit{and} & A & = & \int Z^q_- \, d\widehat{A} = \int f^q(Z_-)\, d\widehat{A} \\ \textit{with} & \widehat{A} & := & \frac{q(q-1)}{2} \langle N^c \rangle + g_q(Y) * \nu^P. \end{array}$$

Moreover, the multiplicative decomposition is $f^q(Z) = \mathcal{E}(\widehat{M})\mathcal{E}(\widehat{A})$ *where* $\mathcal{E}(\widehat{M})$ *is a strictly positive uniformly integrable* P*-martingale. With* $\frac{dR_q}{dP} := \mathcal{E}(\widehat{M})_T$ *the process* $\mathcal{E}(\widehat{A}) = e^{\widehat{A}}$ *is increasing and* R_q*-integrable and*

$$f^q(Q|P) = E_{R_q}\left[\mathcal{E}(\widehat{A})_T\right] = E_{R_q}\left[\exp\left(\int_0^T k(\beta_t, Y_t)\, dt\right)\right] \tag{2.2}$$

where $k(\beta_t, Y_t) := \frac{q(q-1)}{2}\beta_t^* c \beta_t + \int_{\mathbb{R}^d} g_q(Y_t(x))\, K(dx)$.

Proof. See Appendix C. □

Corollary 2.2. *If in Proposition 2.1 we have* $Q^{(\beta,Y)} \in \mathcal{Q}^q \cap \overline{\mathcal{Q}}$*, then*

$$f^q(Q|P) = e^{Tk(\beta,Y)} = \exp\left(T\left(\frac{q(q-1)}{2}\beta^* c\beta + \int_{\mathbb{R}^d} g_q(Y(x))\, K(dx)\right)\right).$$

Proof. Obvious from the definitions of $\overline{\mathcal{Q}}$ and k. □

Remark 2.3. All results are stated for $S = \mathcal{E}(L)$. However, we could equivalently work with $S = e^{\widetilde{L}}$ where $\widetilde{L}$ is a P-Lévy process with P-characteristic triplet $(\widetilde{b}, \widetilde{c}, \widetilde{K})$ since $e^{\widetilde{L}} = \mathcal{E}(L)$ if L has characteristic triplet

$$\begin{array}{rcl} b & = & \widetilde{b} + \frac{1}{2}(\widetilde{c}_{11}, \dots, \widetilde{c}_{dd})^* + \int_{\mathbb{R}^d} (h(e^x - \mathbf{1}) - h(x))\widetilde{K}(dx) \\ c & = & \widetilde{c} \\ K(G) & = & \int_{\mathbb{R}^d} \mathbf{I}_{\{(e^x - \mathbf{1}) \in G\}} \widetilde{K}(dx) \end{array}$$

where $\mathbf{1} = (1, \dots, 1)^*$; this holds by Itô's formula and is stated explicitly in Lemma A.8 in [8]. Analogously, discounting with respect to some numeraire can also be captured by a modification of the characteristic triplet of L; see Corollary A.7 in [8]. ♢

Proposition 2.1 suggests that we have to minimize the expression for $f^q(Q|P)$ obtained in (2.2) over all Girsanov parameters (β, Y) corresponding to some $Q^{(\beta,Y)} \in \mathcal{Q}^q$. However, we will show next that this can be reduced to a much simpler deterministic problem since it suffices to consider time-independent Girsanov parameters. In fact, it is then sufficient to minimize $k(\beta, Y)$ in (2.2) pointwise over (ω, t). The reason for this is the following

PROPOSITION 2.4. *If $\overline{\beta} \in \mathbb{R}^d$ and $\overline{Y} : \mathbb{R}^d \to \mathbb{R}_+$ is a measurable function such that $\overline{Y} > 0$ K-a.e.,*

$$k(\overline{\beta}, \overline{Y}) < \infty \tag{2.3}$$

and the martingale condition

$$(\mathcal{M}) \qquad b + c\overline{\beta} + \int_{\mathbb{R}^d} (x\overline{Y}(x) - h(x))K(dx) = 0 \tag{2.4}$$

holds, then $(\overline{\beta}, \overline{Y})$ are the Girsanov parameters of $\overline{Q} := Q^{(\overline{\beta},\overline{Y})} \in \overline{\mathcal{Q}} \cap \mathcal{Q}^q$.

PROOF. See Appendix C. □

REMARK 2.5. 1. Proposition A.10 in Appendix A explains why we refer to $(\mathcal{M})$ as martingale condition.
2. For all equations like (2.4) we assume implicitly that everything is well-defined, i.e., that $x\overline{Y}(x) - h(x)$ is K-integrable.

◇

THEOREM 2.6. *1. For any $Q \in \mathcal{Q}^q$ there exists $\overline{Q} \in \overline{\mathcal{Q}}$ such that*

$$f^q(\overline{Q}|P) \le f^q(Q|P).$$

2. For every $\varepsilon > 0$ there exists $\overline{Q} \in \overline{\mathcal{Q}}$ such that

$$f^q(\overline{Q}|P) \le \inf_{Q \in \mathcal{M}^e(L)} f^q(Q|P) + \varepsilon.$$

3. If the qMMM Q_q exists, then $Q_q \in \overline{\mathcal{Q}}$.

PROOF. 1. By Propositions 2.1 and A.10 and since $Q \in \mathcal{Q}^q$ there exists $\bar{t} \in [0,T]$ and $\overline{\omega} \in \Omega$ such that $\overline{\beta} := \beta_{\bar{t}}(\overline{\omega})$ and $\overline{Y}(x) := Y_{\bar{t}}(x, \overline{\omega})$ satisfy $\overline{Y}(x) > 0$ K-a.e., $b + c\overline{\beta} + \int_{I\!R^d} (x\overline{Y}(x) - h(x)) = 0$ and

$$e^{Tk(\overline{\beta},\overline{Y})} \leq E_{R_q}\left[\exp\left(\int_0^T k(\beta_t, Y_t)\right)\right] = f^q(Q|P) < \infty.$$

The claim then follows from Proposition 2.4 and Corollary 2.2.

2. Since $\mathcal{Q}^q \neq \emptyset$ and by the definition of the infimum, there exists $Q' \in \mathcal{Q}^q$ such that $f^q(Q'|P) \leq \inf_{Q \in \mathcal{M}^e(L)} f^q(Q|P) + \varepsilon$. Claim 2 then follows from 1. applied to Q'.
3. Since $z \mapsto z^q$ is strictly convex on $I\!R_+$, Q_q is unique. Thus 3. follows immediately from 1. applied to Q_q.

□

The proof of Theorem 2.6 is similar to the ansatz used by Esche and Schweizer in [5] for the minimal entropy martingale measure P_e; the latter is defined as that local martingale measure Q for S which minimizes the relative entropy $E_Q[\log(dQ/dP)]$. However, whereas we define the Girsanov parameters of a measure $\overline{Q}$ with $f^q(\overline{Q}|P) \leq f^q(Q|P)$ via a pointwise minimization of $k(\beta, Y)$ over (ω, t), they average β and Y with respect to an appropriate measure on $\Omega \times [0,T]$; in our setting this would be $dR_q \otimes dt$ with R_q from Proposition 2.1. The advantage of the pointwise minimization is that it ensures that $\overline{Q} \in \mathcal{M}^e(S)$, whereas in [5] Fubini's theorem has to be applied to prove that $\overline{Q}$ is again a local martingale measure. But this requires an additional integrability condition on L which is not satisfied in general. Thus they have to show that there is a dense subset of local martingale measures with this integrability condition and apply additional approximation procedures. This can all be avoided by the pointwise approach.

Theorem 2.6 and Proposition 2.4 even allow to reduce the search for Q_q to finding the solution to the following deterministic optimization problem.

$(\mathcal{P}_q)$: *Find a solution* $(\widehat{\beta}_q, \widehat{Y}_q)$ *to*

$$\textit{minimize} \qquad k(\beta, Y) = \tfrac{q(q-1)}{2}\beta^* c\beta + \int_{I\!R^d} g_q(Y(x))\, K(dx)$$

over

$$\mathcal{A} := \Big\{(\beta, Y) \;\Big|\; \beta \in I\!R^d,\ Y : I\!R^d \to I\!R_+ \text{ measurable}, \; Y > 0 \;\; K\text{-a.e.},\ (\beta, Y) \text{ satisfies } (\mathcal{M})\Big\}.$$

THEOREM 2.7. *1. If the qMMM $Q_q = Q^{(\beta_q, Y_q)}$ exists, then (β_q, Y_q) solves $(\mathcal{P}_q)$, i.e. $(\beta_q, Y_q) = (\widehat{\beta}_q, \widehat{Y}_q)$.*

2. If $(\widehat{\beta}_q, \widehat{Y}_q)$ solves $(\mathcal{P}_q)$, then Q_q exists and has Girsanov parameters $(\widehat{\beta}_q, \widehat{Y}_q)$, i.e., $Q_q = Q^{(\widehat{\beta}_q, \widehat{Y}_q)}$.

PROOF. 1. By 3. of Theorem 2.6 we have $Q_q \in \overline{\mathcal{Q}}$ and Proposition A.10 implies that (β_q, Y_q) satisfies $(\mathcal{M})$ so that $(\beta_q, Y_q) \in \mathcal{A}$. Now suppose there exists $(\beta, Y) \in \mathcal{A}$ such that $k(\beta, Y) < k(\beta_q, Y_q) < \infty$. Then by Proposition 2.4 and Corollary 2.2 there exists $Q := Q^{(\beta,Y)} \in \overline{\mathcal{Q}} \cap \mathcal{Q}^q$ such that

$$f^q(Q|P) = e^{Tk(\beta,Y)} < e^{Tk(\beta_q,Y_q)} = f^q(Q_q|P)$$

which contradicts the definition of Q_q.

2. By Proposition 2.4 $(\widehat{\beta}_q, \widehat{Y}_q)$ defines some $\widehat{Q} := Q^{(\widehat{\beta}_q, \widehat{Y}_q)} \in \mathcal{Q}^q \cap \overline{\mathcal{Q}}$ and by Corollary 2.2 we have $f^q(\widehat{Q}|P) = e^{Tk(\widehat{\beta}_q, \widehat{Y}_q)}$. Suppose there exists $Q' = Q^{(\beta', Y')} \in \mathcal{Q}^q$ such that

$$f^q(Q'|P) < f^q(\widehat{Q}|P). \tag{2.5}$$

By 1. of Theorem 2.6 we may assume that $Q' \in \overline{\mathcal{Q}}$ so that by Proposition A.10 we have $(\beta', Y') \in \mathcal{A}$ and by Corollary 2.2 $f^q(Q'|P) = e^{Tk(\beta', Y')}$. However, then (2.5) implies that $k(\beta', Y') < k(\widehat{\beta}_q, \widehat{Y}_q)$, a contradiction to $(\widehat{\beta}_q, \widehat{Y}_q)$ solving $(\mathcal{P}_q)$. Consequently, $\widehat{Q} = Q_q$. □

REMARK 2.8. Existence results for Q_q can also be found in Bellini and Frittelli [1], but they do not give an explicit formula for Q_q; see also the related article [14] of Kramkov and Schachermayer. ♢

2.2. *Sufficient conditions.* In the previous subsection we presented the deterministic convex optimization problem $(\mathcal{P}_q)$ which has a solution if and only if the qMMM Q_q exists. Moreover, the solution to $(\mathcal{P}_q)$ corresponds to the Girsanov parameters of Q_q and vice versa. Via a formal application of the Kuhn-Tucker theorem to $(\mathcal{P}_q)$, one can obtain the following condition $(\mathcal{C}_q)$; see Theorem 28.3 in [18].

$(\mathcal{C}_q)$: *There exists $\widetilde{\lambda}_q \in \mathbb{R}^d$ with $\widetilde{Y}_q(x) := \left((q-1)\widetilde{\lambda}_q^* x + 1\right)^{\frac{1}{q-1}} > 0$ K-a.e., $\int_{\mathbb{R}^d} g_q(\widetilde{Y}_q(x))\, K(dx) < \infty$ and such that $(\mathcal{M})$ is satisfied for $(\widetilde{\beta}_q, \widetilde{Y}_q)$*

where $\widetilde{\beta}_q := \widetilde{\lambda}_q$.

$(\mathcal{C}_q)$ has the advantage that it can easier be checked in practical applications than $(\mathcal{P}_q)$ and the following theorem proves that $(\mathcal{C}_q)$ implies $(\mathcal{P}_q)$. However, we will show in Section 3 that $(\mathcal{C}_q)$ is in general stronger than $(\mathcal{P}_q)$ and is thus sufficient but not necessary for the existence of Q_q.

THEOREM 2.9. *If* $(\mathcal{C}_q)$ *holds, then* $(\widetilde{\beta}_q, \widetilde{Y}_q)$ *solves* $(\mathcal{P}_q)$*, i.e.,* $(\widetilde{\beta}_q, \widetilde{Y}_q) = (\widehat{\beta}_q, \widehat{Y}_q)$.

PROOF. Note that for a convex function F we have $F(y') - F(y) \geq \frac{d}{dy}F(y)(y'-y)$ and that $\frac{d}{dy}g_q(y) = q(y^{q-1}-1)$. From this and the definition of $(\widetilde{\beta}_q, \widetilde{Y}_q)$ we obtain for any $(\beta', Y') \in \mathcal{A}$ with $k(\beta', Y') < \infty$

$$\begin{aligned} & k(\beta', Y') - k(\widetilde{\beta}_q, \widetilde{Y}_q) \\ \geq\ & \frac{q(q-1)}{2}\left(2\widetilde{\beta}_q^*(c\beta' - c\widetilde{\beta}_q)\right) \\ & + \int_{\mathbb{R}^d} q\left(\widetilde{Y}_q^{q-1}(x) - 1\right)\left(Y'(x) - \widetilde{Y}_q(x)\right) K(dx) \\ =\ & q(q-1)\left(\widetilde{\lambda}_q^*(c\beta' - c\widetilde{\beta}_q) + \int_{\mathbb{R}^d} \widetilde{\lambda}_q^* x\left(Y'(x) - \widetilde{Y}_q(x)\right) K(dx)\right) \\ =\ & 0, \end{aligned}$$

where the last equality holds since $(\widetilde{\beta}_q, \widetilde{Y}_q)$ and (β', Y') satisfy $(\mathcal{M})$. Since for $(\beta', Y') \in \mathcal{A}$ with $k(\beta', Y') = \infty$ obviously $k(\widetilde{\beta}_q, \widetilde{Y}_q) < k(\beta', Y')$ the proof is complete. □

REMARK 2.10.
1. In $(\mathcal{C}_q)$ we assumed $\widetilde{Y}_q(x) > 0$ K-a.e. but not that $\widetilde{Y}_q(x) \geq 0$ for all $x \in \mathbb{R}^d$ which is presumed for $(\widetilde{\beta}_q, \widetilde{Y}_q)$ to be a solution to $(\mathcal{P}_q)$. However, we identify all functionals on $\mathbb{R}^d$ which are K-a.e. equal since they will describe the same probability measure.
2. The assumption $\widetilde{Y}_q(x) > 0$ K-a.e. in $(\mathcal{C}_q)$ looks very restrictive since it holds if and only if $(q-1)\widetilde{\lambda}_q^* x > -1$ K-a.e. However, we assumed that $S = \mathcal{E}(L) > 0$, i.e., that $\Delta L > -1$ so that $\mathrm{supp}(K) \subseteq (-1, \infty)$.
3. Note that for $q = 2$ the condition $\int g_2(\widetilde{Y}_2(x))\, K(dx) < \infty$ is satisfied if and only if $\int (\widetilde{\lambda}_2^* x)^2\, K(dx) < \infty$. In particular, if $d = 1$ (and $\widetilde{\lambda}_2 \neq 0$) this is equivalent to local P-square integrability (and hence P-square integrability) of L; see Proposition II.2.29 of JS.

♢

Condition $(\mathcal{C}_q)$ appears for $q \in (-\infty, 0)$ also in Kallsen [13] and in Goll and Rüschendorf [9]. However, they do not motivate where $(\mathcal{C}_q)$ comes from. Instead they prove by direct calculations that $(\mathcal{C}_q)$ determines the qMMM because the conditions defining an optimal strategy for power utility respectively of an f-projection are satisfied. In a very recent work, Choulli, Stricker and Li [3] state (C_q) for general q, but for the minimal Hellinger martingale measure of order q instead of Q_q. Equivalence of these measures is then shown for discrete time models.

3. The variance minimal martingale measure. In this section we discuss the important special case $q = 2$ and refer to the corresponding qMMM Q_2 as the *variance minimal martingale measure* (VMMM). In Subsection 2.2 we showed that condition $(\mathcal{C}_2)$ is sufficient for the existence of a solution to problem $(\mathcal{P}_2)$ and hence for the existence of Q_2. In the following Subsection 3.1 we present several cases in which $(\mathcal{C}_2)$ is even necessary, provided L satisfies an additional integrability condition. Moreover, in Subsection 3.2 we recall the relation between the VMMM and the very popular *variance optimal signed martingale measure* (VOSMM), which in a Lévy setting equals the *minimal signed martingale measure*. The latter exists if the so called *structure condition* (SC) is satisfied. We clarify the relation between (SC) and $(\mathcal{C}_2)$.

3.1. *Necessary conditions.* As suggested by 3. of Remark 2.10, we assume for the whole subsection that L is (locally) P-square integrable, i.e., we make the

Assumption: $\int_{\mathbb{R}^d} \|x\|^2 \, K(dx) < \infty.$

To state the main theorem of this section we need to introduce the following condition

$(\mathcal{A})$: *The Girsanov parameters (β_2, Y_2) of $Q_2 = Q^{(\beta_2, Y_2)}$ are such that*

$$\mathcal{L}^1(Y_2) := \left\{ \Psi \in L^2(K) \;\middle|\; \int \Psi(x)\, x\, K(dx) = 0, \; |\Psi(x)| \leq aY_2(x) \; K\text{-a.e. for some } a > 0 \right\}$$

is $L^2(K)$-dense in

$$\mathcal{L}^2 := \left\{ \Psi \in L^2(K) \;\middle|\; \int \Psi(x)\, x\, K(dx) = 0 \right\}.$$

THEOREM 3.1. *If the VMMM $Q_2 = Q^{(\beta_2,Y_2)}$ exists, then $(\mathcal{C}_2)$ is satisfied for some $\widetilde{\lambda}_2$ such that $(\widetilde{\lambda}_2, \widetilde{Y}_2) = (\beta_2, Y_2)$ in both of the following cases:*

1. *c is invertible;*
2. *$c = 0$ and $(\mathcal{A})$ holds.*

REMARK 3.2. By Theorem 2.7 we could equivalently state condition $(\mathcal{A})$ and Theorem 3.1 in terms of the solution $(\widehat{\beta_2}, \widehat{Y}_2)$ to $(\mathcal{P}_2)$ instead of the Girsanov parameters (β_2, Y_2) of $Q_2 = Q^{(\beta_2,Y_2)}$. ♢

Before we prove Theorem 3.1, we show that if $c = 0$ and $d = 1$, i.e., if L is one-dimensional and has no Brownian part, then condition $(\mathcal{A})$ is automatically satisfied.

LEMMA 3.3. *If $d = 1$, $c = 0$ and* $\operatorname{supp}(K) \neq \emptyset$ *then $(\mathcal{A})$ holds true.*

PROOF. For an arbitrary $\Psi \in \mathcal{L}^2$ we construct a sequence $(\Psi_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}^1(Y_2)$ converging to Ψ in $L^2(K)$. To this behalf we define for $n \in \mathbb{N}$

$$A_n := \{x \in \mathbb{R} \,|\, |\Psi(x)| \le nY_2(x)\} \quad \text{and} \quad \alpha_n := \int_{\mathbb{R}^d} \Psi(x)\mathbf{I}_{A_n}(x)\, x\, K(dx).$$

By the dominated convergence theorem $\lim_{n\to\infty} \alpha_n = \int \Psi(x)\, x\, K(dx) = 0$; this uses that $\int x^2\, K(dx) < \infty$. Set $\delta(x) := \operatorname{sign}(x)\big(|x| \wedge Y_2(x)\big)$ and note that $|\delta(x)| \le Y_2(x)$ and that $\delta \in L^2(K)$ implies $\delta(x)|x| \in L^1(K)$. Therefore

$$\gamma := \int_{\mathbb{R}} \delta(x)\, x\, K(dx) = \int_{\mathbb{R}} |x|\big(|x| \wedge Y_2(x)\big)\, K(dx) < \infty$$

and $\gamma > 0$ since $\operatorname{supp}(K) \neq \emptyset$ and $K(\{0\}) = 0$ implies that $x\delta(x) > 0$ K-a.e. Let $\Psi_n(x) := \Psi(x)\mathbf{I}_{A_n} - \frac{\alpha_n}{\gamma}\delta(x)$ so that $\Psi_n \in L^2(K)$,

$$|\Psi_n(x)| \le \ |\Psi(x)\mathbf{I}_{A_n}| + \frac{|\alpha_n|}{\gamma}|\delta(x)| \le \left(n + \frac{|\alpha_n|}{\gamma}\right) Y_2(x)$$

and

$$\int_{\mathbb{R}} \Psi_n(x)\, x\, K(dx) = \alpha_n - \frac{\alpha_n}{\gamma} \int_{\mathbb{R}} \delta(x)\, x\, K(dx) = 0$$

so that $\Psi_n \in \mathcal{L}^1(Y_2)$. Moreover, by the dominated convergence theorem and since $\lim_{n\to\infty} \alpha_n = 0$

$$\lim_{n\to\infty} \int_{I\!R} |\Psi(x) - \Psi_n(x)|^2 \, K(dx) \leq \lim_{n\to\infty} 2 \left(\int_{I\!R} |\Psi(x)\mathbf{I}_{A_n^c}|^2 \, K(dx) + \int_{I\!R} \left| \frac{\alpha_n}{\gamma} \, \delta(x) \right|^2 \, K(dx) \right) = 0.$$

This completes the proof. □

Proof of Theorem 3.1. We first recall from Theorem 2.7 that (β_2, Y_2) solves $(\mathcal{P}_2)$. Now let $\phi \in I\!R^d$ and $\Psi \in L^2(K)$ with $|\Psi(x)| \leq aY_2(x)$ K-a.e. for some $a > 0$ and

$$c\phi + \int_{I\!R^d} \Psi(x) x \, K(dx) = 0; \tag{3.1}$$

if $c = 0$ this means that $\Psi \in \mathcal{L}^1(Y_2)$. Note that for $\varepsilon > 0$ small enough $(\beta_2 + \varepsilon\phi, Y_2 + \varepsilon\Psi) \in \mathcal{A}$; w.l.o.g., if the second function is negative on a set of K-measure zero we can modify Ψ there. Since $g_2(y) = (y-1)^2$ and $\int g_2(Y_2(x)) \, K(dx) < \infty$, by Proposition 2.1 we can define

$$H_{\phi,\Psi}(\varepsilon) := (\beta_2 + \varepsilon\phi)^* c \, (\beta_2 + \varepsilon\phi) + \int_{I\!R^d} \big(Y_2(x) + \varepsilon\Psi(x) - 1\big)^2 \, K(dx)$$

and obtain

$$\frac{d}{d\varepsilon} H_{\phi,\Psi}(\varepsilon) = 2 \left(\varepsilon\phi^* c \, \phi + \beta_2^* c \, \phi + \int_{I\!R^d} \Psi(x) \big(Y_2(x) + \varepsilon\Psi(x) - 1\big) \, K(dx) \right).$$

Since (β_2, Y_2) solves $(\mathcal{P}_2)$ we obtain

$$\frac{d}{d\varepsilon} H_{\phi,\Psi}(0) = 2 \left(\beta_2^* c \, \phi + \int_{I\!R^d} \Psi(x) \big(Y_2(x) - 1\big) \, K(dx) \right) = 0. \tag{3.2}$$

We now proceed separately for the two cases of Theorem 3.1 and start with 1. Here (3.2) together with (3.1) yields

$$\int_{I\!R^d} \Big(\beta_2^* x - \big(Y_2(x) - 1\big) \Big) \Psi(x) \, K(dx) = 0. \tag{3.3}$$

and Ψ can be chosen arbitrary under the condition that $|\Psi(x)| \leq aY_2(x)$ K-a.e. for some $a > 0$ because (3.1) can always be satisfied by setting $\phi := -c^{-1} \int x \, \Psi(x) \, K(dx)$. Consequently, (3.3) implies

$$\beta_2^* x - \big(Y_2(x) - 1\big) = 0 \quad K\text{-a.e.} \tag{3.4}$$

In fact, suppose $\beta_2^* x - \big(Y_2(x) - 1\big) > 0$ on a set $A \subseteq \mathbb{R}^d$ with $K(A) > 0$. Then $\widetilde{\Psi}(x) := \Big(\sqrt{\|x\|^2 \wedge 1} \wedge Y_2(x)\Big) \mathbf{I}_A \in L^2(K)$, $\widetilde{\Psi} > 0$ K-a.e., $|\widetilde{\Psi}| \leq Y_2$ and

$$\int_{\mathbb{R}^d} \Big(\beta_2^* x - \big(Y_2(x) - 1\big)\Big) \widetilde{\Psi}(x)\, K(dx) > 0$$

contradicts (3.3). Since (β_2, Y_2) solves $(\mathcal{P}_2)$ we thus obtain from (3.4) that $\widetilde{\lambda}_2 := \beta_2$ satisfies $(\mathcal{C}_2)$. This proves 1. For 2 we introduce the d-dimensional (and hence closed) subspace $\mathcal{L}^0 := \Big\{\alpha^* x \,\Big|\, \alpha \in \mathbb{R}^d\Big\}$ of $L^2(K)$. Note that closedness of $\mathcal{L}^0$ implies that $\mathcal{L}^0 = \big(\mathcal{L}^{0\perp}\big)^{\perp} = \mathcal{L}^{2\perp}$. Therefore (3.2) with $c = 0$ and $(\mathcal{A})$ yield $(Y_2(x) - 1) \in \mathcal{L}^{2\perp} = \mathcal{L}^0$ and hence $(Y_2(x) - 1) = \alpha^* x$ for some $\alpha \in \mathbb{R}^d$. Setting $\widetilde{\lambda}_2 := \alpha$ implies 2 since (β_2, Y_2) solves $(\mathcal{P}_2)$. □

3.2. *Connection to the variance optimal signed martingale measure.* In this subsection we relate the VMMM Q_2 to the *variance optimal signed martingale measure* $\widetilde{P}$ which arises in the mean-variance hedging approach; see [20] for an overview and terminology not explained here. $\widetilde{P}$ is like Q_2 obtained from minimizing the variance of the density dQ/dP. However, the optimization is done over the set of all *signed* local martingale measures for S. If $\widetilde{P}$ is in fact an equivalent measure, then it corresponds to Q_2. Moreover, in a Lévy setting $\widetilde{P}$ corresponds to the *minimal signed martingale measure* $\widehat{P}$ which occurs in the local risk minimizing hedging approach. Even in a more general setting, the *structure condition* (SC) implies the existence of $\widehat{P}$ and an explicit formula for its density is known. We show that in our setting and if L is (locally) P-square integrable, $(\mathcal{C}_2)$ holds if and only if (SC) holds and if $\widehat{P} = \widetilde{P}$ is an equivalent measure. We first introduce (SC) in the Lévy setting. For this, we assume that L is a special semimartingale so that by JS, Corollary II.2.38 and Proposition II.2.29, we have

$$\begin{aligned} L_t &= \Big(L_t^c + x * (\mu^L - \nu^P)_t\Big) + \Big(bt + (x - h(x)) * \nu_t^P\Big) \\ &= \Big(L_t^c + x * (\mu^L - \nu^P)_t\Big) + \Big(b + \int_{\mathbb{R}^d} (x - h(x))\, K(dx)\Big)\, t \\ &=: M_t + \gamma t =: M_t + A_t. \end{aligned}$$

In addition, we need to impose that the local martingale M is locally P-square integrable (and hence P-square integrable), i.e., $\int \|x\|^2\, K(dx) < \infty$.

Then we have

$$\begin{aligned}\langle M\rangle_t &= ct + (xx^*) * \nu_t^P \\ &= \left(c + \int_{\mathbb{R}^d} xx^*\, K(dx)\right) t \\ &=: \sigma t.\end{aligned}$$

(SC) is satisfied if there exists a d-dimensional predictable process λ with

$$A = \int d\langle M\rangle\lambda \quad \text{and} \quad \widehat{K}_T := \int \lambda^*\, d\langle M\rangle\lambda < \infty\,; \tag{3.5}$$

see Definition 1.1 in [2] and Subsection 12.3 in [4] for a related discussion. Since $A_t = \gamma t$ and $\langle M\rangle_t = \sigma t$, (3.5) holds if and only if there exists $\lambda \in \mathbb{R}^d$ such that $\gamma = \sigma\lambda$ or, equivalently,

$$b + \int_{\mathbb{R}^d} (x - h(x))\, K(dx) = \left(c + \int_{\mathbb{R}^d} xx^*\, K(dx)\right) \lambda. \tag{3.6}$$

Under (SC) we can define

$$\widehat{N} := -\int \lambda^*\, dM.$$

If $\widehat{Z} := \mathcal{E}(\widehat{N})$ is a P-martingale, then $\frac{d\widehat{P}}{dP} := \widehat{Z}_T$ defines a signed measure called the minimal signed martingale measure for L. By Proposition 2 of [19], it is a local martingale measure for L in the sense that $\widehat{Z}L$ is a local P-martingale. Note that if $\widehat{Z} > 0$, i.e., if

$$-\lambda^*\Delta M > -1, \tag{3.7}$$

then $\widehat{Z} = \mathcal{E}(\widehat{N})$ is a local martingale and, as in the proof of Proposition A.8, an application of Theorem II.5 of [16] yields that it is a P-martingale, i.e., $\widehat{P} \in \mathcal{M}^e(L)$. Moreover, by Proposition A.9 its Girsanov parameters are

$$\beta := -\lambda \quad \text{and} \quad Y(x) := -\lambda^* x + 1.$$

Under the above assumptions, the *mean-variance tradeoff process*

$$\widehat{K}_t := \int_0^t \lambda^*\, dA_s = \left\langle \int \lambda^*\, dM \right\rangle_t = \lambda^*\sigma\lambda\, t$$

is deterministic. This implies by Theorem 8 of [19] that $\widehat{P}$ is equal to the variance-optimal signed martingale measure $\widetilde{P}$. If we denote the density of $\widetilde{P}$ by $\widetilde{Z}_T$, then $\widetilde{P}$ is defined by the property that

$$E[(\widetilde{Z}_T)^2] \le E[(Z_T)^2]$$

for all P-martingales Z with $Z_0 = 1$ such that ZL is a local P-martingale; the corresponding measures Q with $\frac{dQ}{dP} = Z_T$ are called *signed local martingale measures* for L. Hence, if $\widehat{Z}_T = \widetilde{Z}_T > 0$ so that $\widehat{P} \in \mathcal{M}^e(L) = \mathcal{M}^e(S)$, then $\widehat{P}$ coincides with Q_2.

Having gathered all information required, we next show that (SC) together with the condition that $\widehat{Z}_T > 0$ is actually equivalent to $(\mathcal{C}_2)$. In fact, by (3.6) (SC) with $\widehat{Z}_T > 0$ holds if and only if there exists $\lambda \in I\!R^d$ such that

$$b - c\lambda + \int_{I\!R^d} \Big(x(-\lambda^* x + 1) - h(x) \Big) K(dx) = 0, \quad Y(x) := -\lambda^* x + 1 > 0 \ K\text{-a.e.}$$

With the replacement $\widetilde{\lambda}_2 := \lambda$ this equals $(\mathcal{C}_2)$; note that the assumption $\int \|x\|^2 K(dx) < \infty$ implies $\int g_2(Y(x)) K(dx) < \infty$.

In conclusion, under the assumption that L is a P-square integrable Lévy process we showed that the structure condition (SC) (which yields the existence of $\widehat{P}$ and hence of $\widetilde{P}$) together with the condition that $\widehat{P} \approx P$ is equivalent to $(\mathcal{C}_2)$. The formula for the density of $\widehat{P}$ is then naturally the same as the one which can be deduced from $(\mathcal{C}_2)$.

4. Convergence to the minimal entropy martingale measure. In this section we show that under some technical assumptions, the qMMM Q_q converges in entropy to the minimal entropy martingale measure (MEMM) P_e if q decreases to 1; this generalizes a result of Grandits and Rheinländer from continuous to Lévy processes. In particular, we prove convergence of the Girsanov parameters. At the end of this section we give a general example in which all assumptions we have to impose are satisfied.

We assume that $(\mathcal{C}_q)$ has a solution for all $q \in (1, 1+\varepsilon)$ for some $\varepsilon > 0$. Hence for $q \in (1, 1+\varepsilon)$ there exists $\widetilde{\lambda}_q \in I\!R^d$ such that

$$(q-1)\widetilde{\lambda}_q^* x + 1 > 0 \quad K\text{-a.e.} \tag{4.1}$$

and such that for

$$\Phi(\lambda, q) := b + c\lambda + \int_{I\!R^d} \Big(x((q-1)\lambda^* x + 1)^{\frac{1}{q-1}} - h(x) \Big) K(dx) \tag{4.2}$$

we have $\Phi(\widetilde{\lambda}_q, q) = 0$; in particular $\Phi(\widetilde{\lambda}_q, q)$ is well defined. Recall from Theorem 2.9 that this implies the existence of Q_q and that $Q_q = Q^{(\beta_q, Y_q)}$ where $\beta_q = \widetilde{\lambda}_q$ and $Y_q(x) := \big((q-1)\widetilde{\lambda}_q^* x + 1\big)^{\frac{1}{q-1}}$. A similar existence criterion is known for P_e; see Theorem 3.1 of [7] or Theorem B and Lemma 15 of [5]. In fact, if for

$$\Phi_e(\lambda) := b + c\lambda + \int_{I\!R^d} \Big(x e^{\lambda^* x} - h(x) \Big) K(dx)$$

there exists $\lambda_e \in I\!R^d$ such that $\Phi_e(\lambda_e) = 0$, then P_e exists with Girsanov parameters

$$\beta_e := \lambda_e \quad \text{and} \quad Y_e(x) := e^{\lambda_e^* x}.$$

Since $\lim_{q\searrow 1}\left((q-1)\lambda^* x + 1\right)^{\frac{1}{q-1}} = e^{\lambda^* x}$, we have that for fixed $\lambda \in I\!R^d$ such that $(q'-1)\lambda^* x + 1 > 0$ K-a.e. for $q' := 1+\varepsilon$ and under sufficient integrability conditions that

$$\lim_{q\searrow 1} \Phi(\lambda, q) = \Phi_e(\lambda).$$

Consequently, it is natural to expect that also the solutions $\widetilde{\lambda}_q$ to $\Phi(\lambda, q) = 0$ converge to the solution λ_e to $\Phi_e(\lambda) = 0$. This will be shown next by an application of the implicit function theorem. Therefore we need to assume in addition the existence of some open $D \subseteq I\!R^{d+1}$ such that $\left\{ \left(q, \widetilde{\lambda}_q\right) \middle| \, q \in (1, 1+\varepsilon)\right\}$ is contained in D, Φ is well defined and continuously differentiable on D and $\det\left(\frac{d}{d\lambda}\Phi(q, \widetilde{\lambda}_q)\right) \neq 0$ for all $q \in (1, 1+\varepsilon)$. Then there exists a continuously differentiable function $\lambda(q)$ defined on $(1, 1+\varepsilon)$ such that $\lambda(q) = \widetilde{\lambda}_q$ there. Finally we need to assume that $\widetilde{\lambda}_1 := \lim_{q\searrow 1} \lambda(q)$ exists and that $\lim_{q\searrow 1} \Phi(q, \lambda(q)) = \Phi_e(\widetilde{\lambda}_1)$; this holds if $\lambda(\,.\,)$ is bounded on $(1, 1+\varepsilon)$ and if we can interchange limit and integration in (4.2). Since $\Phi(q, \lambda(q)) \equiv 0$ we then have $\Phi_e(\widetilde{\lambda}_1) = 0$ and hence $\widetilde{\lambda}_1 = \lambda_e$ as required. Obviously, then also

$$\lim_{q\searrow 1} \beta_q = \lim_{q\searrow 1} \lambda(q) = \lambda_e = \beta_e \tag{4.3}$$

$$\lim_{q\searrow 1} Y_q(x) = \lim_{q\searrow 1} \left((q-1)\lambda^*(q)x + 1\right)^{\frac{1}{q-1}} = e^{\lambda_e^* x} = Y_e(x) \quad K\text{-a.e.} \tag{4.4}$$

In particular, we have convergence of the Lévy measure of Q_q to that of P_e in the sense that

$$\lim_{q\searrow 1} K^{\text{qMMM}}(dx) = \lim_{q\searrow 1} Y_q(x)\, K(dx) = Y_e(x)\, K(dx) = K^{\text{MEMM}}(dx);$$

see Proposition A.1 below. Finally we show that Q_q converges to P_e in entropy, i.e., that the relative entropy of Q_q with respect to P_e

$$H(Q_q|P_e) := E_{Q_q}\left[\log \frac{dQ_q}{dP_e}\right] = E_{Q_q}\left[\log Z_T^{Q_q} - \log Z_T^{P_e}\right]$$

converges to 0 if q decreases to 1. From Proposition A.3, the formula for the stochastic exponential, and Proposition II.1.28 of JS together with Lemma

B.4, 3. of Theorem 2.6 and Corollary 2.2 we obtain

$$\begin{aligned}
&\log Z_T^{Q_q}\\
&= \beta_q^* L_T^c + (Y_q - 1) * (\mu^L - \nu^P)_T - \frac{1}{2} T \beta_q^* c \beta_q + \big(\log Y_q - (Y_q - 1)\big) * \mu_T^L\\
&= \beta_q^* L_T^c + (\log Y_q) * (\mu^L - \nu^P)_T - \frac{1}{2} T \beta_q^* c \beta_q + \big(\log Y_q - (Y_q - 1)\big) * \nu_T^P.
\end{aligned}$$

Applying the same arguments and replacing Theorem 2.6 and Corollary 2.2 by [5], Theorem A and Lemma 12, we obtain for $\log Z_T^{P_e}$ the same expression with (β_q, Y_q) replaced by (β_e, Y_e). Thus

$$\begin{aligned}
\log \frac{dQ_q}{dP_e} &= (\beta_q^* - \beta_e^*) L_T^c + (\log Y_q - \log Y_e) * (\mu^L - \nu^P)_T\\
&\quad - \frac{1}{2} T \big(\beta_q^* c \beta_q - \beta_e^* c \beta_e\big) + \big(\log Y_q - Y_q - \log Y_e + Y_e\big) * \nu_T^P.
\end{aligned}$$

Recall from Girsanov's theorem that $\nu^{Q_q} := Y_q \nu^P$ is the Q_q-compensator of μ^L and that $\widetilde{L}$ with $\widetilde{L}_t := L_t^c - t c \beta_q$ is a Q_q-martingale. If $(\log Y_q - \log Y_e) * \nu^{Q_q}$ is the Q_q-compensator of $(\log Y_q - \log Y_e) * \mu^L$, then

$$\begin{aligned}
\log \frac{dQ_q}{dP_e} &= (\beta_q^* - \beta_e^*) \widetilde{L}_T + (\log Y_q - \log Y_e) * (\mu^L - \nu^{Q_q})_T\\
&\quad - \frac{1}{2} T \big(\beta_q^* c \beta_q - \beta_e^* c \beta_e\big) + T \big(\beta_q^* - \beta_e^*\big) c \beta_q\\
&\quad + \big((\log Y_q - \log Y_e) Y_q - (Y_q - Y_e)\big) * \nu_T^P;
\end{aligned}$$

see Proposition II.1.28 of JS. The first two terms on the RHS are local Q_q-martingales and Q_q-Lévy processes and thus Q_q-martingales so that

$$\begin{aligned}
H(Q_q|P_e) &= -\frac{1}{2} T \big(\beta_q^* c \beta_q - \beta_e^* c \beta_e\big) + \big(\beta_q^* - \beta_e^*\big) c \beta_q T\\
&\quad + E_{Q_q}\Big[\big((\log Y_q - \log Y_e) Y_q - (Y_q - Y_e)\big) * \nu_T^P\Big].
\end{aligned}$$

Consequently, if integration and limit are interchangeable, then (4.3) and (4.4) imply that

$$\lim_{q \searrow 1} H(Q_q|P_e) = 0.$$

Example: In the following example, all assumptions of this section are satisfied. Let $K(dx) = f(x)\,dx$ where f is a bounded density such that $\mathrm{supp}(K) \subseteq (-1, \ell)$ with $0 < \ell < \infty$. Moreover, let L be of dimension one and if L has no Brownian part, i.e., $c = 0$, then let it have positive and negative jump heights, i.e., $K\big((-1,0)\big) > 0$ and $K\big((0,\ell)\big) > 0$. We show that there exists $\varepsilon > 0$ such that $(\mathcal{C}_q)$ has a solution $\widetilde{\lambda}_q$ for all $q \in (1, 1+\varepsilon)$ and that we can take

$$D := D(\varepsilon) := \Big\{(q,\lambda)\Big| q \in (1, 1+\varepsilon),\, q \in \big(\gamma_1(q), \gamma_2(q)\big)\Big\}$$

with $\gamma_1(q) := -\frac{1}{\ell(q-1)}$ and $\gamma_2(q) := \frac{1}{q-1}$. All integrability conditions of Section 4 are then satisfied due to boundedness of f and of $\mathrm{supp}(K)$; note that $y \mapsto y \log y$ is bounded from below and that $Y^e(x) = \exp(\lambda_e^* x)$ is K-a.e. bounded away from 0. For $q \in (1,2)$ and $\lambda \in (\gamma_1(q), \gamma_2(q))$ condition (4.1) is satisfied and

$$\begin{aligned}
\frac{d}{d\lambda}\Phi(\lambda,q) &= c + \int_{-1}^{\ell} \left(x^2\big((q-1)\lambda x + 1\big)^{\frac{2-q}{q-1}} \right) f(x)\, dx \\
&\geq \begin{cases} c + \int\limits_{-1}^{0} x^2 f(x)\, dx =: \delta_1 > 0 & \text{if } \lambda \in \big(\gamma_1(q), 0\big], \\ c + \int\limits_{0}^{\ell} x^2 f(x)\, dx =: \delta_2 > 0 & \text{if } \lambda \in \big[0, \gamma_2(q)\big). \end{cases}
\end{aligned}$$

Thus $\frac{d}{d\lambda}\Phi(\lambda,q) \geq \delta := \min\{\delta_1, \delta_2\} > 0$. Let

$$b_0 := \Phi(\lambda, 0) = \Phi_e(\lambda) = b + \int_{-1}^{\ell} \big(x - h(x)\big) f(x)\, d(x)$$

and note that $\lim_{q \searrow 1} \gamma_1(q) = -\infty$ and $\lim_{q \searrow 1} \gamma_2(q) = \infty$. If $b_0 < 0$, we hence can find $\varepsilon_1 > 0$ such that $\gamma_2(q) > |b_0|/\delta$ for all $q \in (1, 1+\varepsilon_1)$. Then for all $q \in (1, 1+\varepsilon_1)$ there exists a solution $\widetilde{\lambda}_q \in (0, |b_0|/\delta) \subseteq \big(0, \gamma_2(q)\big)$ to $\Phi(\lambda, q) = 0$ and we can take $D = D(\varepsilon_1)$; note in addition that $q \mapsto \widetilde{\lambda}_q$ is bounded. Analogously, if $b_0 > 0$, we select $\varepsilon_2 > 0$ such that $|\gamma_1(q)| > b_0/\delta$ for all $q \in (1, 1+\varepsilon_2)$ which implies for these q the existence of a solution $\widetilde{\lambda}_q \in (-b_0/\delta, 0) \subseteq (\gamma_1(q), 0)$ to $\Phi(\lambda, q) = 0$ and that we can take $D = D(\varepsilon_2)$. Finally, $b_0 = 0$ is a trivial case, since we then have $\Phi(0, q) = 0$ for all $q > 1$ so that $\beta_q = 0$ and $Y_q = 1$, i.e., $Q_q = P = P_e$. This concludes the example.

Acknowledgements: Part of this work was completed during during a visit of MJ to Graduate School of Economics, Nagoya City University. She expresses her gratitude for the kind invitation and financial support. SK wishes to thank M. Schweizer for many helpful discussions and suggestions and YM wants to thank H. Kunita for many valuable discussions.

November 23, 2006

APPENDIX A: CHANGE OF MEASURE AND LÉVY PROCESSES

In this section we gather the required results on changes of measure and Lévy processes. In particular, we give conditions under which two processes are the Girsanov parameters of an equivalent local martingale measure. For unexplained notation we refer to [12], abbreviated JS in the sequel.

We fix a filtered probability space $(\Omega, \mathcal{F}, I\!\!F, P)$ with $I\!\!F = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfying the usual conditions under P and $\mathcal{F}_0$ trivial. For a semimartingale X and a probability measure $Q \approx P$, we denote by μ^X the random measure associated with the jumps of X and by ν^Q the predictable Q-compensator of μ^X. If μ is a random measure and W is sufficiently integrable, we denote by $W * \mu$ the integral process of W with respect to μ. Moreover, we work throughout with the truncation function $h(x) := x\mathbf{I}_{\{\|x\|\leq 1\}}$; the results do not depend on this particular choice of h. By $\mathbf{P}$ we denote the predictable σ-field on $\Omega\times[0,T]$ and by (B, C, ν) the P-characteristics of the semimartingale X with respect to h. As in Proposition II.2.9 of JS, we can and do always choose a version of the form

$$B = \int b\,dA, \qquad C = \int c\,dA, \qquad \nu(\omega;\, dx, dt) = K_{\omega,t}(dx)\,dA_t(\omega), \tag{A.1}$$

where A is a real-valued predictable increasing locally integrable process, b an $I\!\!R^d$-valued predictable process, c a predictable process with values in the set of all symmetric nonnegative definite $d \times d$-matrices, and $K_{\omega,t}(dx)$ a transition kernel from $(\Omega \times [0,T], \mathbf{P})$ into $(I\!\!R^d, \mathcal{B}^d)$ with $K_{\omega,t}(\{0\}) = 0$ and $\int_{I\!\!R^d}(1 \wedge \|x\|^2)K_{\omega,t}(dx) \leq 1$ for all $t \leq T$. We denote by $\|\,.\,\|$ the usual norm on $I\!\!R^d$.

We now turn to the description of absolutely continuous probability measures. The following Girsanov type result shows that any $Q \ll P$ can be described by two parameters β and Y.

Proposition A.1. (Theorem III.3.24 of JS) *Let X be a semimartingale with P-characteristics (B^P, C^P, ν^P) and denote by c, A the processes from (A.1). For any probability measure $Q \ll P$, there exist a $\mathbf{P}\otimes\mathcal{B}^d$-measurable function $Y \geq 0$ and a predictable $I\!\!R^d$-valued process β satisfying*

$$\|(Y-1)h\| * \nu^P_T + \int_0^T \|c_s\beta_s\|\,dA_s + \int_0^T \beta^*_s c_s\beta_s\,dA_s < \infty \quad Q\text{-a.s.}$$

and such that the Q-characteristics (B^Q, c^Q, ν^Q) of X are given by

$$B_t^Q = B_t^P + \int_0^t c_s\beta_s\,dA_s + ((Y-1)h) * \nu_t^P,$$
$$C_t^Q = C_t^P,$$
$$\nu^Q(dx,dt) = Y_t(x)\,\nu^P(dx,dt).$$

We call β and Y the Girsanov parameters *of Q (with respect to P, relative to X) and write $Q = Q^{(\beta,Y)}$ to emphasize the dependence.*

REMARK A.2. 1. In Proposition A.1 we have $Y(x) > 0$ $dP \otimes dt$-a.e. for K-a.e. x if and only if $Q \approx P$.

2. Note that β and Y are not unique. In fact, Y is unique only ν^P-a.e., and for fixed c and A we have A-a.e. uniqueness only for $c\beta$. In the whole article we fix a Lévy process L and express the Girsanov parameters of any $Q \ll P$ relative to L. We then identify all versions of Girsanov parameters (β, Y) which describe the same Q. In particular, if we say that the Girsanov parameters (β, Y) of Q are time-independent, we mean that there exists one version with this property.

$\diamondsuit$

A.1. Lévy processes. Let $Q \approx P$ and $L = (L_t)_{0\le t\le T}$ be an $\mathbb{F}$-adapted stochastic process with RCLL paths and $L_0 = 0$. Then L is called a $(Q, \mathbb{F})$-*Lévy process* if for all $s \le t \le T$, the increment $L_t - L_s$ is independent of $\mathcal{F}_s$ under Q and has a distribution which depends on $t-s$ only. Recall that a Lévy process is a Feller process, so that $\mathbb{F}^{L,Q}$, the Q-augmentation of the filtration generated by L, automatically satisfies the usual conditions under Q. If $Q = P$, we even sometimes drop the mention of P, i.e., refer to L simply as a Lévy process and write $\mathbb{F}^L$. In particular, if $Q = P$ and $\mathbb{F} = \mathbb{F}^L$ for quantities depending on P and L we often do not write this dependence explicitly; this is done, e.g., for Girsanov parameters. We will use frequently that for $Q \approx P$, every $(Q, \mathbb{F})$-Lévy process is an $\mathbb{F}$-semimartingale and a $(Q, \mathbb{F})$-martingale if and only if it is a $(Q, \mathbb{F})$-local martingale; see [11], Theorem 11.46.

Another important fact is that Lévy processes have the weak predictable representation property; see JS, Theorem III.4.34. That is, if $\mathbb{F} = \mathbb{F}^L$, then every local P-martingale (starting in zero) is the sum of an integral with respect to the continuous martingale part L^c and an integral with respect to the compensated jump measure. This allows to give an explicit formula for the density process of any probability measure $Q \approx P$ with Girsanov parameters β and Y.

PROPOSITION A.3. (Proposition 3 of [5]) *Let L be a P-Lévy process and $\mathbb{F} = \mathbb{F}^L$. If $Q \approx P$ with Girsanov parameters (β, Y), the density process of Q with respect to P is given by $Z^Q = \mathcal{E}(N^Q)$ with*

$$N_t^Q = \int_0^t \beta_s^* \, dL_s^c + (Y-1) * (\mu^L - \nu^P)_t.$$

REMARK A.4. We frequently use that in the setting of Proposition A.3 and for $f : (-1, \infty) \to \mathbb{R}$ sufficiently integrable we have $\sum_{s \le t} f(\Delta N_s^Q) = f(Y-1) * \mu_t^L$. ♢

In (A.1) we have introduced an integral version for the characteristics of a semimartingale. It is well known that a Lévy process can be characterized by the particular structure of its characteristics.

LEMMA A.5. (Corollary II.4.19 of JS) *Let $Q \approx P$ and L be an $(Q, \mathbb{F})$-semimartingale. Then L is a $(Q, \mathbb{F})$-Lévy process if and only if there exists a version of its Q-characteristics such that*

$$B_t^Q(\omega) = b^Q t, \quad C_t^Q(\omega) = c^Q t, \quad \nu^Q(\omega; dx, dt) = K^Q(dx)\, dt \tag{A.2}$$

where $b^Q \in \mathbb{R}^d$, c^Q is a symmetric non-negative definite $d \times d$-matrix, K^Q is a positive measure on $\mathbb{R}^d$ that integrates $(\|x\|^2 \wedge 1)$ and satisfies $K^Q(\{0\}) = 0$. We call (b^Q, c^Q, K^Q) the Lévy characteristics *of L (with respect to Q).*

REMARK A.6. For a P-Lévy process we drop the mention of P and denote the Lévy characteristics by (b, c, K). Moreover, without further mention we always use the notation

$$\nu^P(dx, dt) = K(dx)\, dt.$$

♢

As an immediate consequence of Girsanov's theorem and Lemma A.5, we obtain for any $(P, \mathbb{F})$-Lévy process L the following well-known characterization of the set of all probability measures $Q \approx P$ under which L is a $(Q, \mathbb{F})$-Lévy process.

COROLLARY A.7. *Let L be an $(P, \mathbb{F})$-Lévy process and $Q \approx P$ with Girsanov parameters β and Y. Then L is a $(Q, \mathbb{F})$-Lévy process if and only if β and $Y(x)$ are $dP \otimes dt$-a.e. time-independent and deterministic for K-a.e. $x \in \mathbb{R}^d$.*

A.2. Change of measure. In the previous subsections, we have described for any $Q \approx P$ the corresponding Girsanov parameters. Now we want to start with arbitrary predictable processes β and Y and give conditions under which they define a probability measure $Q \approx P$ and can be identified as the Girsanov parameters of Q. Obviously, β and Y have to satisfy some integrability conditions. In view of our aim to minimize $f^q(Q|P) = E\big[(Z_T^Q)^q\big]$ for $q \in I = (-\infty, 0) \cup (1, \infty)$ over some set of probability measures $Q \approx P$, we impose integrability conditions on β and Y which naturally arise in the computation of $f^q(Q|P)$; see Proposition 2.1. We formulate them in terms of the strictly convex function $g_q : (0, \infty) \to \mathbb{R}_+$:

$$g_p(y) := y^q - 1 - q(y - 1).$$

PROPOSITION A.8. *Let L be a P-Lévy process with Lévy characteristics (b, c, K), $\mathbb{F} = \mathbb{F}^L$, $q \in I$, β a predictable process and $Y \geq 0$ a $\mathbf{P} \otimes \mathcal{B}^d$-measurable function with $y(x) > 0$ K-a.e. If*

$$\int_{\mathbb{R}^d} g_q(Y_s(x))\, K(dx) \leq \text{const} \qquad dP \otimes dt\text{-a.e.,} \tag{A.3}$$

then $Y - 1$ is integrable with respect to $\mu^L - \nu^P$. If in addition

$$\beta_s^* c \beta_s \leq \text{const} \quad dP \otimes dt\text{-a.e.,} \tag{A.4}$$

then $Z := \mathcal{E}(N)$ with

$$N_t = \int_0^t \beta_s^*\, dL_s^c + (Y - 1) * (\mu^L - \nu^P)_t \tag{A.5}$$

is a strictly positive P-martingale.

PROOF. The integrability of $Y - 1$ with respect to $\mu^L - \nu^P$ follows from Lemma B.1 together with Theorem II.1.33 d) in JS. Thus by (A.4) N is a local martingale and in addition quasi-left-continuous, so that by Theorem II.5 in [16] $\mathcal{E}(N)$ is a martingale if the predictable compensator of $\langle N^c \rangle_\cdot +$

$\sum_{s\leq \cdot} \left((\Delta N_s)^2 \wedge |\Delta N|_s\right)$ is bounded; note that for Theorem II.5 of [16] it suffices if N is a local martingale. In addition, $\mathcal{E}(N)$ is strictly positive since $Y > 0$ implies that $\Delta N > -1$ so that it only remains to show boundedness of the compensator. For $\langle N^c \rangle = \int \beta_t^* c \beta_t \, dt$ which is already the predictable compensator of itself, the claim is trivial by (A.4). The jump term can be rewritten as

$$\sum_{s\leq t} \Big((\Delta N_s)^2 \wedge |\Delta N_s|\Big) = \Big((Y-1)^2 \wedge |Y-1|\Big) * \mu_t^L. \tag{A.6}$$

Since N is in particular a special semimartingale, (A.6) defines by Propositions II.1.28 and II.2.29 a) of JS a locally integrable process. Also by Proposition II.1.28, the latter has $(Y-1)^2 \wedge |Y-1| * \nu^P$ as predictable P-compensator. This compensator is then bounded thanks to Lemma B.3 and assumption (A.3). This finishes the proof. □

The following result now allows to identify a priori given β and Y with the Girsanov parameters of the measure Q defined via $Z = \mathcal{E}(N)$, where N is constructed from β and Y via (A.5).

Proposition A.9. (Proposition 7 of ES) *Let L be a P-Lévy process with Lévy characteristics (b, c, K) and $\mathbb{F} = \mathbb{F}^L$. Let β be a predictable process, integrable with respect to L^c, and $Y > 0$ a predictable function such that $Y - 1$ is integrable with respect to $\mu^L - \nu^P$, and define $N := \int \beta_s^* \, dL_s^c + (Y-1) * (\mu^L - \nu^P)$. If there exists a probability measure $Q \approx P$ with density process $Z^Q = Z := \mathcal{E}(N)$, then β and Y are the Girsanov parameters of Q.*

We finish this section with a result which gives condition on the Girsanov parameters (β, Y) for $Q^{(\beta,Y)} \approx P$ to be an equivalent local martingale measure for a P-Lévy processes L, i.e., for $Q \in \mathcal{M}^e(L)$.

Proposition A.10. (Theorem 3.1 of [15]) *Let L be a P-Lévy process with Lévy characteristics (b, c, K), $\mathbb{F} = \mathbb{F}^L$ and $Q \approx P$ with Girsanov parameters β and Y. Then $Q \in \mathcal{M}^e(L)$ if and only if*

$$b + c\beta_t + \int_{\mathbb{R}^d} (xY_t(x) - h(x)) \, K(dx) = 0 \quad dQ \otimes dt\text{-}a.e. \tag{A.7}$$

Condition (A.7) is called the martingale condition *for L.*

APPENDIX B: AUXILIARY RESULTS

This section contains some simple auxiliary results.

LEMMA B.1. *Fix $q \in I$. Then there exists $c = c(q) > 0$ such that*

$$(1-\sqrt{y})^2 \le c g_q(y) \qquad \text{for all } \; y > 0.$$

PROOF. We need to find $c > 0$ such that $f(y) := c g_q(y) - (1-\sqrt{y})^2$ is non-negative on $(0,\infty)$. For $q < 0$ we can take $c = -\frac{1}{q}$ since $f(1) = 0$, $\frac{d}{dy} f(y) < 0$ on $(0,1)$ and $\frac{d}{dy} f(y) > 0$ on $(1,\infty)$. For $q > 1$ take $c > \frac{1}{q-1} > \frac{1}{2q(q-1)}$ and define $\overline{y} := \big(2cq(q-1)\big)^{\frac{1}{1/2-q}}$. Calculating $\frac{d^2}{dy^2} f$ yields strict concavity of f on $(0,\overline{y})$ and strict convexity on $(\overline{y},\infty)$. Moreover, $\overline{y} < 1$, $\frac{d}{dy} f(1) = 0$ and $f(1) = 0$. Since f is continuous it is thus non-negative on $(0,\infty)$ if $f(0) \ge 0$ which holds true by the choice of c. □

LEMMA B.2. *Fix $q \in I$ and $\overline{y} > 1$. Then there exists a constant $C = C(\overline{y}, q) > 0$ such that for all $c \ge C$*

$$(y-1)^2 \le c g_q(y) \qquad \text{for all } \; y \in (0,\overline{y}].$$

PROOF. Define $f(y) := c g_q(y) - (y-1)^2$. For $q < 2$ let $C := \frac{2\overline{y}^{2-1}}{q(q-1)}$. Then f is convex on $(0,\overline{y})$ with minimum in $y = 1$ where $f(1) = 0$ so that f is non-negative on $(0,\overline{y}]$. For $q \ge 2$ set $C := \frac{2}{q}$ so that $\frac{d}{dy} f(y) \le 0$ on $(0,1)$ and $\frac{d}{dy} f(y) \ge 0$ on $(1,\overline{y}]$. Since $f(1) = 0$ we have $f(y) \ge 0$ on $(0,\overline{y}]$. □

LEMMA B.3. *For $q \in I$ there exists $C = C(q) > 0$ such that*

$$(y-1)^2 \wedge |y-1| \le C g_q(y) \quad \text{for all } y > 0.$$

PROOF. Lemma B.2 with $\overline{y} = 2$ implies the claim for $0 \le y \le 2$. For $y > 2$, note that $(y-1)^2 \ge |y-1| = y-1$, and define $f(y) := C g_q(y) - (y-1)$ with $C \ge \max\left\{-\frac{2}{q}, \frac{1}{g_q(2)}\right\}$ for $q < 0$ and $C \ge \max\left\{\frac{1}{q(2^{q-1}-1)}, \frac{1}{g_q(2)}\right\}$ for $q > 1$. Then f is increasing on $[2,\infty)$ and $f(2) \ge 0$. □

LEMMA B.4. *For $q \in I$ and $y > 0$ we have*

$$\log y - (y-1) \le g_q(y) \quad \text{and} \quad \log y - (y-1) \le y \log y - (y-1).$$

PROOF. Both $y \mapsto g_q(y) - \big(\log y - (y-1)\big)$ and $y \mapsto y \log y - (y-1) - \big(\log y - (y-1)\big)$ are strictly convex functions on $\mathbb{R}_+$. Their unique minimum is in $y = 1$ where they are equal to 0 so that they are non-negative on $\mathbb{R}_+$. □

APPENDIX C: OMITTED PROOFS

This section contains the proof omitted in Section 2.

PROOF OF PROPOSITION 2.1. Itô's formula applied to $Z = Z^Q = \mathcal{E}(N)$ yields

$$\begin{aligned} Z_t^q &= \int_0^t Z_{s-}^q \left(q\, dN_s + \frac{q(q-1)}{2}\, d\langle N^c \rangle_s \right) \\ &\quad + \sum_{s \le t} Z_{s-}^q \left((\Delta N_s + 1)^q - q(\Delta N_s + 1) - 1 + q \right). \end{aligned}$$

Recall from Proposition A.3 the expression for N and note that $\langle N^c \rangle = \int \beta_t^* c \beta_t \, dt$ and N are locally P-integrable and, since $Q \in \mathcal{Q}^q$, so is Z^q. Thus also $\sum_{s \le t} Z_{s-}^q \left((\Delta N_s + 1)^q - q(\Delta N_s + 1) - 1 + q \right)$ and

$$\begin{aligned} \sum_{s \le t} \left((\Delta N_s + 1)^q - q(\Delta N_s + 1) - 1 + q \right) &= \big(Y^q - 1 - q(Y-1) \big) * \mu^L \\ &= g_q(Y) * \mu^L \end{aligned}$$

are locally P-integrable. Since g_q is nonnegative, Proposition II.1.28 of JS then implies that the predictable compensator of $g_q(Y) * \mu^L$ is $g_q(Y) * \nu^P$. Moreover,

$$(Y^q - 1 - q(Y-1)) * (\mu^L - \nu^P) + q(Y-1) * (\mu^L - \nu^P) = (Y^q - 1) * (\mu^L - \nu^P)$$

since both sides are local martingales having the same jumps; see Definition II.1.27 in JS. From this and the formula for N from Proposition A.3 we obtain the canonical decomposition

$$\begin{aligned}
dZ^q &= Z^q_- \Big(q\, dN^c + d\left((Y^q - 1) * (\mu^L - \nu^P) \right) \\
&\qquad + \frac{q(q-1)}{2} d\langle N^c \rangle + d\left((Y^q - 1 - q(Y-1)) * \nu^P \right) \Big) \\
&= Z^q_- \left(d\widehat{M} + d\widehat{A} \right) \\
&= d\mathcal{E}\left(\widehat{M} + \widehat{A} \right) \\
&= d\left(\mathcal{E}(\widehat{M})\mathcal{E}(\widehat{A}) \right), \qquad \text{(C.1)}
\end{aligned}$$

where the last equality holds by Yor's formula since $\widehat{A}$ is of finite variation and continuous so that $[\widehat{M}, \widehat{A}] \equiv 0$. Moreover, $Q \in \mathcal{Q}^q$ implies that Z^q is a positive submartingale and thus of class (D) since $0 \le Z^q_\tau \le E\left[Z^q_T | \mathcal{F}_\tau \right]$ for all stopping times $\tau \le T$. Since $\widehat{A} \ge 0$ we have $\mathcal{E}(\widehat{A}) = e^{\widehat{A}} \ge 1$ so that (C.1) implies that $\mathcal{E}(\widehat{M})$ is a local P-martingale of class (D) and thus a martingale; this uses that $\mathcal{E}(\widehat{M})$ is positive since $\Delta\widehat{M} > -1$ P-a.s. because $Y(x) > 0$ K-a.s. implies that $Y^q(x) - 1 > -1$ K-a.s. Moreover, (C.1) then implies the R^p-integrability of $\mathcal{E}(\widehat{A})$ and the strict positivity of $\mathcal{E}(\widehat{M})$. This completes the proof. □

Proof of Proposition 2.4. Propositions A.8, A.9 and A.10 imply that $(\overline{\beta}, \overline{Y})$ are the Girsanov parameters of some $\overline{Q} = Q^{(\overline{\beta}, \overline{Y})} \in \mathcal{M}^e(L) = \mathcal{M}^e(S)$. It remains to show that $\overline{Q} \in \mathcal{Q}^q$ which can be done as in the proof of Proposition 2.1 by an application of Itô's formula to obtain the canonical decomposition and in particular that $f^q(Z^{\overline{Q}}_T) = e^{\widehat{A}_T(\overline{Q})}\mathcal{E}\big(\widehat{M}(\overline{Q})\big)_T$. The only difference in the proof is the way one obtains that $g_q(\overline{Y}) * \mu^L$ is locally P-integrable; this cannot be done as before since we do not know that $\overline{Q} \in \mathcal{Q}^q$. However, since g_q is non-negative, we obtain from (2.3) that $g_q(\overline{Y}) * \nu^P$ is locally P-integrable and this is by Proposition II.1.28 of JS equivalent to local P-integrability of $g_q(\overline{Y}) * \mu^L$. Thus it only remains to show that $f^q(Q|P) = E\Big[e^{\widehat{A}_T(\overline{Q})}\mathcal{E}\big(\widehat{M}(\overline{Q})\big)_T \Big] < \infty$. This holds true since $\Delta\widehat{M}(\overline{Q}) > -1$ implies that $\mathcal{E}\big(\widehat{M}(\overline{Q})\big)$ is a P-supermartingale and since $\widehat{A}_T(\overline{Q})$ is deterministic and finite. □

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Equipe d'analyse et probabilités
Université d'Evry Val d'Essonne
Rue du Père Jarlan
91025 Evry Cedex
France
E-mail: monique.jeanblanc@univ-evry.fr

Vienna University of Technology
Finanz- u. Versicherungsmathematik
Wiedner Hauptstrasse 8 /105-1 FAM
1040 Vienna
Austria
E-mail: kloeppel@fam.tuwien.ac.at

November 23, 2006

Graduate School of Economics
Nagoya City University
Mizuhochou Mizuhoku
Nagoya 8501
Japan
E-mail: y-miya@econ.nagoya-cu.ac.jp