Evaluation of the Scale Risk

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1 Introduction

We study the evaluation problem of the scale risk. The method we have adopted is the risk sensitive value measure (RSVM) method, which have been introduced in [8].

This method is developed originally for the project evaluation. Even so this method can be applied to many evaluation problems in finance. For example we can apply this method to the scale risk evaluation problems.

In this paper we overview the idea of the scale risk evaluation problem. For the details, see [8] and etc.

2 Risk-Sensitive Value Measure (RSVM)

We give the definition of the Risk-Sensitive Value Measure (RSVM) and summarize the properties of this measure.

2.1 Definition of the Risk-Sensitive Value Measure

Definition 1 (Risk sensitive value measure (RSVM)) Let \( X \) be a linear space of random variables, then the risk sensitive value measure (RSVM) on \( X \) is the following functional defined on \( X \)

\[
U^{(\alpha)}(X) = -\frac{1}{\alpha} \log E[e^{-\alpha X}], \quad (\alpha > 0),
\]

where \( \alpha \) is the risk avertion parameter.

Remark 1 In the above definition, \( X \) is supposed to be the random present value of a cash flow or a return of some asset.

2.2 Properties of the Risk-Sensitive Value Measure

We first remark the following facts.

Proposition 1 (i) The following approximation formula holds true:

\[
U^{(\alpha)}(X) = E[X] - \frac{1}{2} \alpha V[X] + \cdots.
\]

(ii) If \( X \) is Gaussian, then it holds that

\[
U^{(\alpha)}(X) = E[X] - \frac{1}{2} \alpha V[X].
\]
2.2.1 Concave Monetary Value Measure

**Definition 2 (concave monetary value measure)** A function \( v(\cdot) \) defined on a linear space \( X \) of random variables is called a concave monetary value measure (or concave monetary utility function) on \( X \) if it satisfies the following conditions:

(i) (Normalization) : \( v(0) = 0 \),
(ii) (Monetary property) : \( v(X + m) = v(X) + m \), where \( m \) is non-random,  
( Remark: (i) + (ii) \( \rightarrow \) \( v(m) = m \) ),
(iii) (Monotonicity) : If \( X \geq Y \), then \( v(X) \geq v(Y) \),
(iv) (Concavity) : \( v(\lambda X + (1 - \lambda)Y) \geq \lambda v(X) + (1 - \lambda)v(Y) \) for \( 0 \leq \lambda \leq 1 \),
(v) (Law invariance) : \( v(X) = v(Y) \) whenever \( \text{law}(X) = \text{law}(Y) \).

**Remark 2** We don’t require the following positive homogeneity property:

(vi) (Positive Homogeneity): \( \forall \lambda \in R^+, \ v(\lambda X) = \lambda v(X) \).

We next notice an important property of a concave monetary value measure.

**Proposition 2 (global concavity)** A concave monetary value measure \( v(\cdot) \) satisfies the following global concavity condition.

(iv) ’ (global concavity) :
\[
v(\lambda X + (1 - \lambda)Y) \leq \lambda v(X) + (1 - \lambda)v(Y) \quad \text{for } \lambda \leq 0 \text{ or } \lambda \geq 1
\]

**Proposition 3** Let \( v(\cdot) \) be a concave monetary value measure. Then, for a fixed pair \((X, Y)\), \( \psi_{X,Y}(\lambda) = v(\lambda X + (1 - \lambda)Y) \) is a concave function of \( \lambda \).

Setting \( Y = 0 \) in this proposition, we obtain the following result:

**Corollary 1** Let \( v(\cdot) \) be a concave monetary value measure. Then \( \psi_X(\lambda) = v(\lambda X) \) is a concave function of \( \lambda \) and \( \psi_X(0) = 0 \).

From this corollary we obtain the following concept of “Optimal Scale.”

**[Optimal Scale]**

Let \( v(\cdot) \) be a concave monetary value measure, and assume that \( v(X_0) > 0 \) for some fixed random variable \( X_0 \). If \( v(\lambda X_0), \lambda > 0, \) is an upper bounded function of \( \lambda \), then we can find the maximum point \( \lambda \). This value \( \lambda \) is the optimal scale of \( X_0 \).

2.2.2 Utility Indifference Value

For a utility indifference value we obtain the following result:

**Proposition 4** Let \( u(x) \) be a utility function defined on \((−\infty, \infty)\) and satisfy the usual properties of a utility function. Then the indifference value \( v(X) \) determined by the following equation
\[
E[u(-v(X) + X)] = u(0) = 0
\]
(2.4)
is a concave monetary value measure.
Remark 3 An indifference value does not satisfy the following positive homogeneity condition in general.

(Positive Homogeneity): \( \forall \lambda \in R^+, \ v(\lambda X) = \lambda v(X) \).

Proposition 5 \( U^{(\alpha)}(X) \) is the indifference value of the exponential utility function:

\[
u_\alpha(x) = \frac{1}{\alpha} \left( 1 - e^{-\alpha x} \right), \quad -\infty < x < \infty \quad (\alpha > 0).
\] (2.5)

Corollary 2 \( U^{(\alpha)}(X) \) is a concave monetary value measure.

Corollary 3 \( U^{(\alpha)}(\lambda X) \) is a concave function of \( \lambda \).

2.2.3 Optimal Scale

From the fact that \( U^{(\alpha)}(\lambda X) \) is a concave function of \( \lambda \), we can discuss the optimal scale of the investment, and we obtain the following result:

Proposition 6 Assume that the moment generation function of \( X \) converges and that the following conditions satisfied,

\[
E[X] > 0, \quad P(X < 0) > 0.
\] (2.6)

Then it holds that

(i) When \( \lambda > 0 \) is small, \( U^{(\alpha)}(\lambda X) > 0 \), and

\[
\lim_{\lambda \to \infty} U^{(\alpha)}(\lambda X) = -\infty.
\] (2.7)

(ii) The optimal scale \( \lambda_{\text{opt}} \) is

\[
\lambda_{\text{opt}} = \frac{C_X}{\alpha}, \quad \alpha > 0,
\] (2.8)

where \( C_X \) is a solution of \( E[Xe^{-C_X X}] = 0 \).

2.2.4 Independence-Additivity Property

Definition 3 (Independence-Additivity) If a value measure \( v(\cdot) \) satisfies e) (independence-additivity): \( v(X + Y) = v(X) + v(Y) \) if \( X \) and \( Y \) are independent, then \( v(\cdot) \) is said to have the independence-additivity property.

We can suppose that this property is desirable for the project evaluation functional, and the following proposition is easily proved.

Proposition 7 An indifference value determined from an exponential utility function has the independence-additivity property.

The converse of this proposition is known.
Proposition 8 Let \( v(x) \) be an indifference value determined by a utility function \( u(x) \) which is of \( C^2 \)-class, increasing, concave, and normalized such as \( u(0) = 0, u'(0) = 1, \) and \( u''(0) = \alpha. \) Then, if \( v(x) \) has the independence-additivity property, \( u(x) \) is of the following form

\[
\begin{align*}
u(x) &= u_\alpha(x) = \frac{1}{\alpha} \left( 1 - e^{-\alpha x} \right). \quad (2.9)
\end{align*}
\]

2.3 Good Points of Risk Sensitive Value Measure

(1) The RSVM is a concave monetary value measure.
(2) The RSVM is the utility indifference value of the exponential utility function, and it has a risk aversion parameter \( \alpha. \)
(3) The optimal scale of a project can be discussed.
(4) The RSVM has the independence-additivity property, and the RSVM is the almost only one which has this property in the set of all utility indifference values.
(5) The dynamic RSVM has the time-consistency property, and the RSVM is the almost only one which has this property in the set of all utility indifference values.

3 Evaluation of the Scale Risk

3.1 What is the Scale Risk

Let \( X \) be a return for an investment of \( I. \) We suppose that the return for the investment \( \lambda I \) is \( \lambda X. \) Assume that \( E[X] > 0 \) and \( P(X < 0) > 0. \) If \( \lambda(> 0) \) is small then the investment \( \lambda I \) may be positively valued. But if \( \lambda \) is very large, then a very big loss may happen and so the investment \( \lambda I \) may be negatively valued. This is the “scale risk.”

3.2 Numerical Example

Let \( X, Y, Z \) be random variables whose distributions are

\[
\begin{align*}
P(X = -10) &= 0.02, \quad P(X = 4) = 0.5, \quad P(X = 8) = 0.48 \quad (3.1) \\
E[X] &= 5.64, \quad V[X] = 8.9104, \quad (3.2) \\
P(Y = -2) &= 0.15, \quad P(Y = 4) = 0.7, \quad P(Y = 10) = 0.15 \quad (3.3) \\
E[Y] &= 4.00, \quad V[Y] = 10.8000, \quad (3.4) \\
P(Z = -1) &= 0.3, \quad P(Z = 4) = 0.6, \quad P(Z = 16) = 0.1 \quad (3.5) \\
E[Z] &= 3.70, \quad V[Z] = 21.8100. \quad (3.6)
\end{align*}
\]

From the scale risk point of view, \( X \) has a big scale risk, \( Z \) has a less scale risk and \( Y \) is between \( X \) and \( Z. \) Remak here also that

\[
\begin{align*}
E[X] &> E[Y] > E[Z] \quad (3.7) \\
V[X] &< V[Y] < V[Z] \quad (3.8)
\end{align*}
\]
We calculate the values of $\lambda X$, $\lambda Y$ and $\lambda Z$. In the following table,

\[
MV_X(\lambda) = E[\lambda X] - \frac{1}{2}\alpha V[\lambda X], \quad RSV M_X(\lambda) = U^{(\alpha)}(\lambda X),
\]

\[
MV_Y(\lambda) = E[\lambda Y] - \frac{1}{2}\alpha V[\lambda Y], \quad RSV M_Y(\lambda) = U^{(\alpha)}(\lambda Y),
\]

\[
MV_Z(\lambda) = E[\lambda Z] - \frac{1}{2}\alpha V[\lambda Z], \quad RSV M_Z(\lambda) = U^{(\alpha)}(\lambda Z),
\]

where $MV_X$ is the mean variance value of $X$.

\[\alpha = 0.05\]

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<th>$MV_X$</th>
<th>$RSV M_X$</th>
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From the above table we can see that the RSVM is a desirable value measure which contains the evaluation of scale risk.

4 Hedging of the Scale Risk

A numerical example
Let $X$ and $W$ be given as follows,

$$P(\{\omega_1\}) = 0.02, \quad P(\{\omega_2\}) = 0.5, \quad P(\{\omega_3\}) = 0.48,$$

(4.1) \quad X(\omega_1) = -10, X(\omega_2) = 4, X(\omega_3) = 8; \quad E[X] = 5.64, V[X] = 8.9104, \quad (4.2)

$$W(\omega_1) = 10, W(\omega_2) = -1, W(\omega_3) = -1; \quad E[W] = -0.7800, V[W] = 2.3716. \quad (4.3)$$

(The distribution of $X$ is same as before.)

Then we obtain

$$U(0.05)(X) = 5.381304 > 0, \quad U(0.05)(10X) = -22.268194 < 0$$

(4.4) \quad U(0.05)(W) = -0.8301 < 0, \quad U(0.05)(10W) = -9.5976 < 0. \quad (4.5)

So, $X$ may be carried out but $10X$, $W$ and $10W$ are not carried out.

On the other hand, we obtain the following results,

$$U(0.05)(X + W) = 4.7498 > 0, \quad U(0.05)(10X + 10W) = 38.4748 > 0.$$

(4.6)

Therefore, both $X + W$ and $10X + 10W$ may be carried out. This means that $W$ or $10W$ are valueless, but we can hedge the scale risk of $10X$ by the use of $10W$.

### 5 Inner Rate of Risk Aversion (IRRA)

#### 5.1 Definition of the Inner Rate of Risk Aversion (IRRA)

**Definition 4** Let $X$ be an asset. Then a solution $\alpha$ of the following equation

$$U^{(\alpha)}(X) = 0 \quad (5.1)$$

is called the inner rate of risk aversion (IRRA) of $X$, and denoted by $\alpha_0(X)$.

**Remark 4** The larger $\alpha_0(X)$ is, the smaller the risk of $X$ is. So the IRRA can be a rating index of assets.

#### 5.2 Existence of the IRRA

For the existence of IRRA, we obtain the following result:

**Proposition 9** Assume that the moment generation function of a random variable $X$ converges, and the following conditions satisfied,

$$E[X] > 0 \quad \text{and} \quad P(X < 0) > 0. \quad (5.2)$$

Then the IRRA $\alpha_0(X)$ of $X$ exists and is unique.
6 Concluding Remarks

The books and articles relating to this paper are listed in the References. ([1, 2, 3, 5, 6, 7, 8, 9, 10, 14, 15])

[Problems to which the Risk-Sensitive Value Measure Method can be Applied]
(1) Project evaluation.
(2) Evaluation of financial (or real) assets.
(3) Evaluation of big projects (energy or resources exploitation).
(4) Evaluation of research projects.
(5) Evaluation of the intellectual property.
(6) Evaluation of the credit risk.
(7) Evaluation of a portfolio.
(8) Evaluation of a company.

The papers, [4], [11], [12], [13] are relating to those applications.

References


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