Volatility Smile/Smirk Properties of [GLP & MEMM] Models

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Abstract

The [GLP(Geometric Lévy Process) & MEMM(Minimal Entropy Martingale Measure)] pricing model was first introduced in [13] as one of the pricing models for the incomplete market. We first explain the structure of this model, and next we investigate the volatility smile/smirk properties of this model by the use of computer simulation method.

1   Introduction

It is well-known that the implied volatility surface has smile or smirk properties in the real markets, and this property is called volatility smile/smirk property (or smile/skew property). This fact suggests us that the construction of a new option pricing model, which possesses this smile/smirk property, is inevitable.

Several kinds of models have been proposed and investigated. The [GLP & MEMM] pricing model is one of them, and it is known that this model have many good properties as an option pricing model for the incomplete market. (See [13] and [15]). In this paper we investigate the volatility smile/smirk properties of the [GLP & MEMM] pricing models by the use of computer simulation method.

In §2 we survey the volatility smile/smirk problems. In §3 we explain the [GLP & MEMM] model briefly and in §4 we give several examples of it. In §5 we see that the [GLP & MEMM] model possesses the volatility smile/smirk properties in various forms. Finally in §6 we discuss the calibration results of of Nikkei 225 options by the [GLP & MEMM] models.

The results obtained in this paper show us that the [GLP & MEMM] model is a very strong candidate for the new model which should have the volatility smile/smirk property and that this model can be applied to the calibration problems.

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2   Volatility Smile/Smirk Problems

2.1   Historical Volatility and Implied Volatility for Black-Scholes Model

The Black-Scholes model is the special case of GLP model with no jump part, namely the process is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \tag{2.1}$$

or in the form of stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{2.2}$$

where μ is called the drift parameter and σ is called the volatility parameter.

The theoretical B-S price of the European call option C_K with the strike price K and the fixed maturity T is given by the following formula

$$C_K = S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2), \tag{2.3}$$

where $\Phi(d)$ is the normal distribution function and

$$d_1 = \frac{\log \frac{S_0}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\log \frac{S_0}{K} + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2.4}$$

Under the above setting, the historical volatility of the process is the estimated value of σ based on the sequential data of the price process S_t. We denote it by $\widehat{\sigma}$. On the other hand the implied volatility is defined as what follows. Suppose that the market price of the European call option with the strike K, say $C_K^{(m)}$, were given. Then the value of σ which satisfies the following equation

$$S_0 \Phi(d_1) - e^{-rT} K \Phi(d_2) = C_K^{(m)}, \tag{2.5}$$

is the implied volatility, and this value is denoted by $\sigma_K^{(im)}$. We remark here that the implied volatility $\sigma_K^{(im)}$ depends on the strike value K, and on the contrary the historical volatility $\widehat{\sigma}$ does not depend on K.

2.2   Volatility Smile/Smirk Properties of Market Option Prices

We first consider the case where the market value of options obey to the Black-Scholes model, and so the market price $C_K^{(m)}$ is equal to the theoretical B-S price C_K of (2.3). In this case the solution of the equation (2.5) is equal to the original σ and it holds true that $\sigma_K^{(im)} = \sigma = constant$. This means that if the market obeys exactly to the Black-Scholes model, then the implied volatility $\sigma_K^{(im)}$ should be equal to the historical volatility $\widehat{\sigma}\ (= \sigma)$.

But in the real world this is not true. It is well-known that the implied volatility is not equal to the historical volatility, and the implied volatility curve $\sigma_K^{(im)}$ is sometimes a convex function of K, and sometimes the combination of convex part and concave part. These properties are so-called volatility smile or smirk properties.

The Figure 1, 2, 3, and Table 1, 2 illustrate this situation, where a new variable moneyness $(= \frac{K}{S_0})$ is introduced.
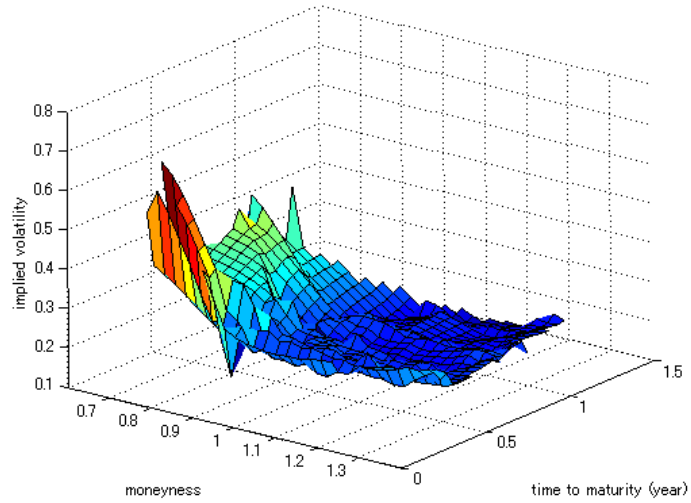
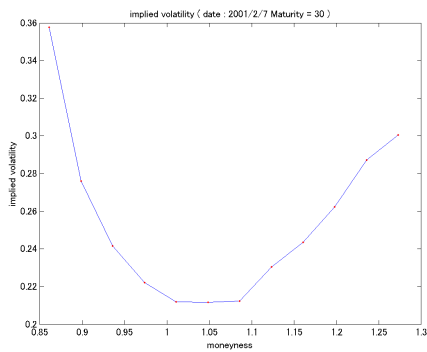Figure 1: Implied volatility surface for Nikkei 225 index options. (July-2002, time to maturity is 20-day or more)

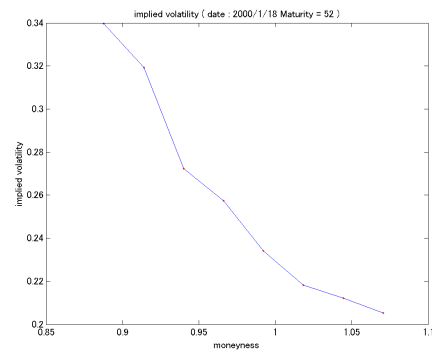Figure 2: 07/02/2001, time to maturity = 30-day, volatility smile

Figure 3: 18/01/2000, time to maturity = 52-day, volatility smirk

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Table 1: Historical Volatility vs Implied Volatility

To consider the volatility smile/smirk properties of market option prices, we use the data of Nikkei 225 call options traded at OSE (Osaka Stock Exchange), whose time to maturity is less than or equal to 40-days and whose trading volume is more than 50. In this case, we have obtained such a result that the Nikkei 225 call option has the volatility smile property for each year.

err ratio: mean during 1-year of (implied volatility - historical volatility) / historical volatility. The historical volatility is estimated from the log returns of recent 245 days.

	1998	1999	2000	2001	2002	2003	2004
moneyness $\leq$ 0.9	1.2795	1.6432	1.7748	2.0539	2.2575	2.8476	3.0841
0.9 < moneyness $\leq$ 0.95	0.4660	0.4358	0.6954	0.5861	0.3623	0.3323	0.3922
0.95 < moneyness $\leq$ 1.05	0.0587	-0.0576	0.1467	0.1688	0.0711	0.0668	-0.0356
1.05 < moneyness $\leq$ 1.10	0.0494	-0.0746	0.1344	0.1670	0.1125	0.1348	0.0162
1.10 < moneyness	0.1148	-0.0250	0.2756	0.3477	0.2639	0.2949	0.1201

Table 2: Historical Volatility vs Implied Volatility

To consider the volatility smile/smirk properties of market option prices, we use the data of Nikkei 225 call options traded at OSE (Osaka Stock Exchange), whose time to maturity is more than 40-days and whose trading volume is more than 50. In this case, we have obtained such a result that the Nikkei 225 call option has the volatility smirk property for each year.

err ratio: mean during 1-year of (implied volatility - historical volatility) / historical volatility. The historical volatility is estimated from the log returns of recent 245 days.

	1998	1999	2000	2001	2002	2003	2004
moneyness $\leq$ 0.9	0.0939	0.2004	0.3314	0.3468	0.4950	0.4169	0.4272
0.9 < moneyness $\leq$ 0.95	-0.0607	-0.0325	0.1202	0.0503	-0.0390	-0.0324	-0.0821
0.95 < moneyness $\leq$ 1.05	-0.0450	-0.1071	0.0542	0.0105	-0.0459	-0.0562	-0.1037
1.05 < moneyness $\leq$ 1.10	-0.0684	-0.1347	0.0253	-0.0205	-0.0649	-0.0545	-0.1103
1.10 < moneyness	-0.0216	-0.1463	0.0865	-0.0208	-0.0461	-0.0209	-0.1042

2.3 Volatility Smile/Smirk Properties of Models

Based on the above facts, new models, which have the volatility smile/smirk property, are required. A pricing model is said to have the volatility smile property or volatility smirk property if the implied volatility function for the theoretical option prices of that model has the volatility smile or smirk property. Let C_K^* be the theoretical option price of the new model and let $\sigma_K^{(im)*}$ be the solution of

$$S_0\Phi(d_1) - e^{-rT}K\Phi(d_2) = C_K^*. \tag{2.6}$$

Then $\sigma_K^{(im)*}$ is the implied volatility related to the new model. If $\sigma_K^{(im)*}$ has the volatility smile property, then the model is said to have the volatility smile property, and if $\sigma_K^{(im)*}$ has the volatility smirk property, then the model is said to have the volatility smirk property.

3 [GLP & MEMM] Pricing Model

The [GLP & MEMM] Pricing Model is such a model:
(A) The price process S_t is a geometric Lévy process (GLP).
(B) The price of an option X is defined by $e^{-rT}E_{P^*}[X]$, where P^* is the minimal entropy martingale measure (MEMM).

This model was first introduced in [13], and the properties of this model are summarised in [15].

3.1 Geometric Lévy Process (GLP)

The price process S_t of a stock is assumed to be defined as what follows. We suppose that a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{F}_t, 0 \geq t \geq T\}$ are given, and that the price process $S_t = S_0 e^{Z_t}$ of a stock is defined on this probability space and given in the form

$$S_t = S_0 e^{Z_t}, \quad 0 \leq t \leq T, \tag{3.1}$$

where Z_t is a Lévy process. We call such a process S_t the geometric Lévy precess (GLP), and we denote the generating triplet of Z_t by $(\sigma_Z^2, \nu_Z(dx), b_Z)$ or simply by $(\sigma^2, \nu(dx), b)$.

3.2 Minimal Entropy Martingale Measure (MEMM)

We first give the definition of the MEMM.

Definition 1 (minimal entropy martingale measure (MEMM)) *If an equivalent martingale measure P^* satisfies*

$$H(P^*|P) \leq H(Q|P) \qquad \forall Q: \quad equivalent \quad martingale \quad measure, \tag{3.2}$$

then P^* is called the minimal entropy martingale measure (MEMM) of S_t. Where $H(Q|P)$ is the relative entropy of Q with respect to P

$$H(Q|P) = \left\{ \begin{array}{ll} \int_\Omega \log[\frac{dQ}{dP}]dQ, & if \quad Q \ll P, \\ \infty, & otherwise, \end{array} \right\}.$$ (3.3)

3.3 Sufficient Conditions for the Existence of the MEMM

The existence problem of the MEMM of geometric Lévy processes has been studies in [12], [3] and [13], and finally those results are generalized in [7] as the following form.

Theorem 1 (Fujiwara-Miyahara [7, Theorem 3.1]) *Suppose that the following condition* **(C)** *holds*

Condition (C) *There exists $\theta^* \in R$ which satisfies the following conditions :*

$$\textbf{(C)}_\textbf{1} \qquad \int_{\{x>1\}} e^x e^{\theta^*(e^x-1)}\nu(dx) < \infty,$$ (3.4)

$$\textbf{(C)}_\textbf{2} \quad b + (\tfrac{1}{2} + \theta^*)\sigma^2 + \int_{\{|x|>1\}}(e^x-1)e^{\theta^*(e^x-1)}\,\nu(dx)$$
$$+ \int_{\{|x|\leq1\}}\left((e^x-1)e^{\theta^*(e^x-1)} - x\right)\nu(dx) = r.$$ (3.5)

Then the probability measure P^* is well defined and it holds that
(i)(MEMM): P^* *is the MEMM of S_t.*
(ii)(Lévy process): Z_t *is also a Lévy process w.r.t. P^*, and the generating triplet (A^*, ν^*, b^*) of Z_t under P^* is*

$$A^* = \sigma^2,$$ (3.6)
$$\nu^*(dx) = e^{\theta^*(e^x-1)}\nu(dx),$$ (3.7)
$$b^* = b + \theta^*\sigma^2 + \int_{R\setminus\{0\}} xI_{\{|x|\leq1\}}d(\nu^*-\nu).$$ (3.8)

3.4 Prices of European Call Options

We investigate the European call option. The price of European call option is

$$C(S_0, K, T) = e^{-rT}E_{P^*}[(S_T - K)^+].$$ (3.9)

It is known that this value is computed as follows.

The characteristic function $\phi_t^*(u)$ of Z_t under the MEMM $P^*$is

$$\phi_t^*(u) = \phi_{Z_t}^*(u) = E_{P^*}[e^{iuZ_t}] = \exp(\psi_t^*(u)) = \exp(t\psi^*(u)), \quad i = \sqrt{-1},$$ (3.10)

where $\psi^*(u) = \psi_1^*(u)$.

Set

$$\zeta(v; S_0, T) = S_0 \frac{e^{-rT}\phi_T^*(v-i) - e^{ivrT}}{iv(1+iv)}$$ (3.11)

and

$$\tilde{c}(k; S_0, T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikv} \zeta(v; S_0, T) dv \tag{3.12}$$

Then the price of the European call option $C(S_0, K, T)$ is computed as

$$C(S_0, K, T) = \tilde{c}(\log(K/S_0); S_0, T) + (S_0 - e^{-rT}K)^+. \tag{3.13}$$

(See [15].) This method is used in §5 for the computation of the theoretical option prices.

4 Examples of [GLP & MEMM] Pricing Model

In this section we see several examples of [GLP & MEMM] Pricing Models. To do this, we have to check the existence of the MEMM, i.e. we have to examine that the given geometric Lévy process $S_t = S_0 \exp Z_t$ satisfies the Condition (C). Set

$$\begin{aligned}
f(\theta) &= b + (\frac{1}{2} + \theta)\sigma^2 + \int_{\{|x|>1\}} (e^x - 1)e^{\theta(e^x - 1)} \nu(dx) \\
&\quad + \int_{\{|x|\le1\}} \left((e^x - 1)e^{\theta(e^x - 1)} - x \right) \nu(dx).
\end{aligned} \tag{4.1}$$

Then the condition $(C)_2$ is equivalent to that θ^* is the solution of

$$f(\theta) = r. \tag{4.2}$$

4.1 [Geometric Stable Process & MEMM] Model

We consider the stable model. Suppose that Z_t is a stable process and let $(0, \nu(dx), b)$ be its generating triplet. The Lévy measure is

$$\nu(dx) = c_1 I_{\{x<0\}} |x|^{-(\alpha+1)} dx + c_2 I_{\{x>0\}} |x|^{-(\alpha+1)} dx, \tag{4.3}$$

where $0 < \alpha < 2$ and we assume that

$$c_1 > 0, \quad c_2 > 0. \tag{4.4}$$

It is shown that the equation (4.2) has a unique solution θ^* and that θ^* is negative. (See [7] or [16].) So the geometric stable process model is an example of the [GLP & MEMM] Pricing Model.

4.2 [Geometric CGMY Process & MEMM] Model

The Lévy measure of the CGMY process is

$$\nu(dx) = C \left(I_{\{x<0\}} \exp(-G|x|) + I_{\{x>0\}} \exp(-M|x|) \right) |x|^{-(1+Y)} dx, \tag{4.5}$$

where $C > 0, G \ge 0, M \ge 0, Y < 2$ (see [1]). If $Y \le 0$, then $G > 0$ and $M > 0$ are assumed. We mention here that the case $Y = 0$ is the VG process case, and the case $G = M = 0$ and $0 < Y < 2$ is the symmetric stable process case. In the sequel we assume that $G, M > 0$.

For this model the following results are obtained (see [16]).

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Proposition 1 (1) *If $M \leq 1$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is negative.*

(2) If $M > 1$ and $f(0) \geq r$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.

(3) If $M > 1$ and $f(0) < r$, then the equation $f(\theta) = r$ has no solution.

4.3 [Geometric Variance Gamma Process & MEMM] Model

The Lévy measure of Variance Gamma process is of the following form (see [8]).

$$\nu(dx) = C \left(I_{\{x<0\}} \exp(-c_1|x|) + I_{\{x>0\}} \exp(-c_2|x|) \right) |x|^{-1} dx, \tag{4.6}$$

where C, c_1, c_2 are positive constants.

The following results are obtained (see[7] or [16]).

Proposition 2 (1) *If $c_2 \leq 1$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is negative.*

(2) If $c_2 > 1$ and $f(0) \geq r$, then the equation $f(\theta) = r$ has a unique solution θ^*, and the solution is non-positive.

(3) If $c_2 > 1$ and $f(0) < r$, then the equation $f(\theta) = r$ has no solution.

5 Volatility Smile/Smirk Properties of [GLP & MEMM] Models: Simulation Results.

For the models described in the previous sections we have carried on the simulation analysis, and we have obtained many results which support that our model has the volatility smile/smirk property. The following figures of implied surface are typical examples where the volatility smile/smirk property can be seen.
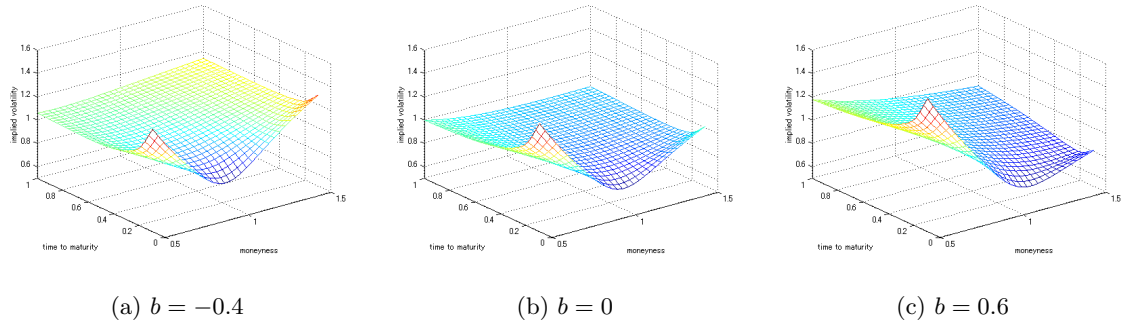
5.1 [Geometric Stable Process & MEMM] Model

(a) $b = -0.4$ (b) $b = 0$ (c) $b = 0.6$

Figure 4: [G-Stable Process & MEMM] Model ($\alpha = 1.2, c_1 = 0.2, c_2 = 0.2$)

(a) $\alpha = 0.8$ (b) $\alpha = 1.4$ (c) $\alpha = 1.8$
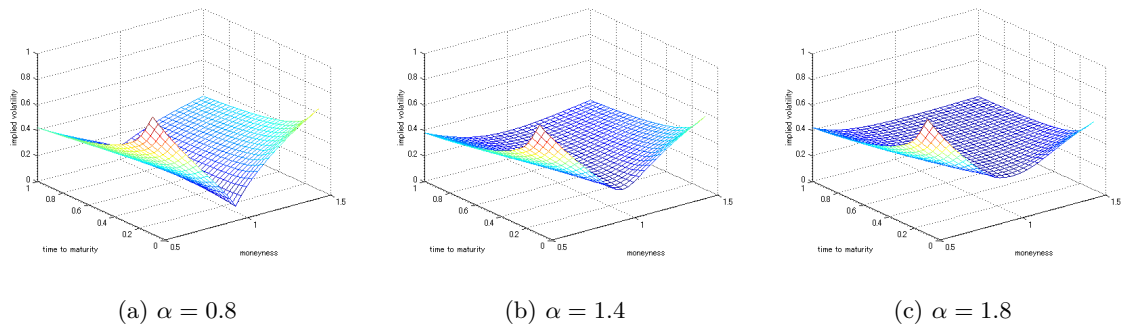
Figure 5: [G-Stable Process & MEMM] Model ($b = 0, c_1 = 0.01, c_2 = 0.01$)

(a) $c_1 = 0.005, c_2 = 0.095$ (b) $c_1 = 0.095, c_2 = 0.005$ (c) $c_1 = 0.1, c_2 = 0.1$
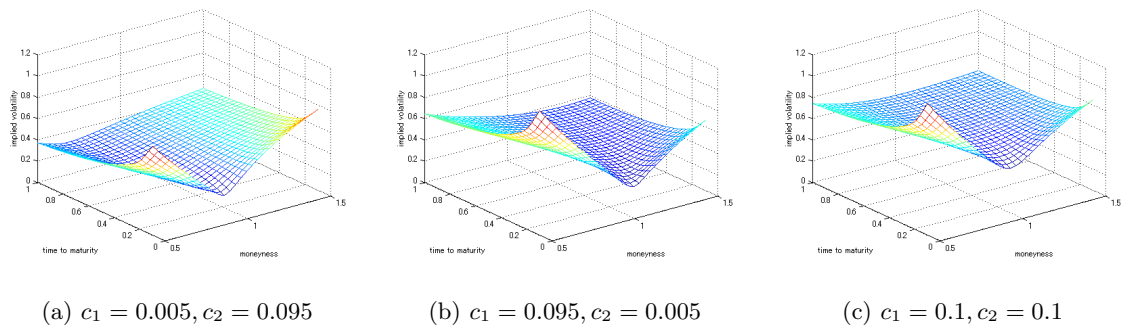
Figure 6: [G-Stable Process & MEMM] Model ($b = 0, \alpha = 1.2$)

5.2 [Geometric CGMY Process & MEMM] Model

(a) $b = -0.1$

(b) $b = 0$

(c) $b = 0.2$
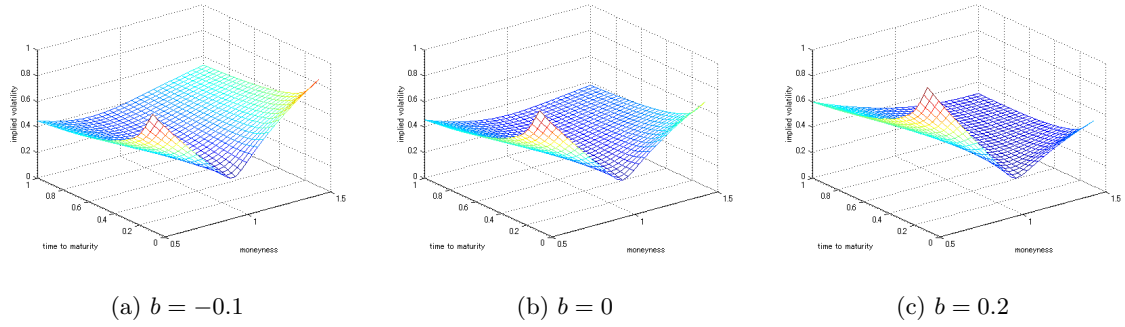
Figure 7: [G-CGMY Process & MEMM] Model ($C = 0.05, G = 0.5, M = 0.5, Y = 1.2$)

(a) $C = 0.05$

(b) $C = 0.1$

(c) $C = 0.3$
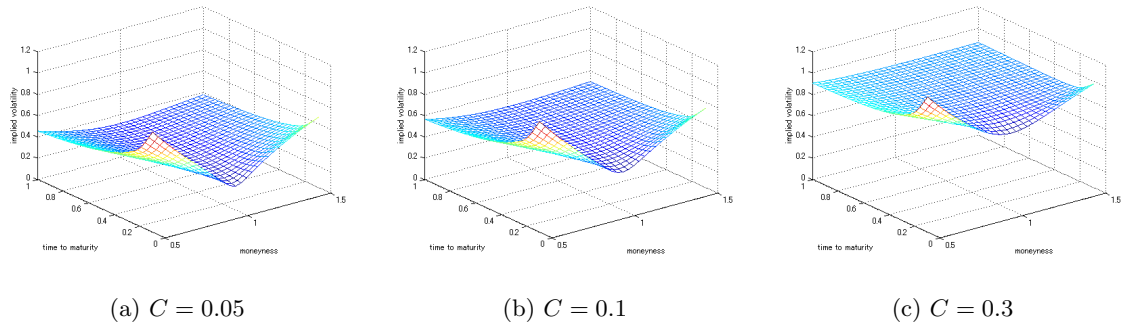
Figure 8: [G-CGMY Process & MEMM] Model ($b = 0, G = 0.5, M = 0.5, Y = 1.2$)

(a) $G = 5, M = 0.05$

(b) $G = 0.05, M = 0.05$

(c) $G = 0.05, M = 3$
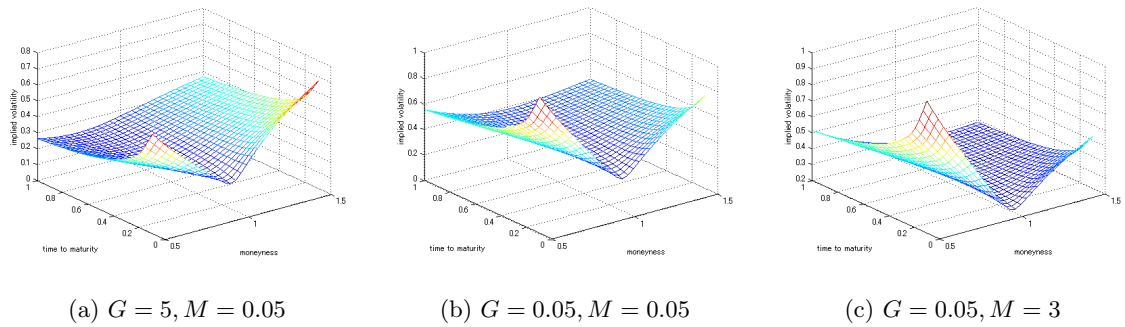
Figure 9: [G-CGMY Process & MEMM] Model ($b = 0, C = 0.05, Y = 1.2$)

(a) $Y = 0.5$ (b) $Y = 1$ (c) $Y = 1.5$
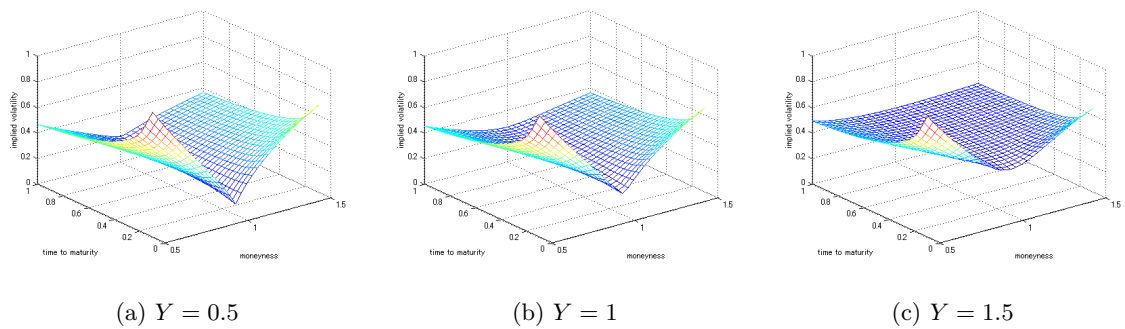
Figure 10: [G-CGMY Process & MEMM] Model ($b = 0, C = 0.05, G = 0.5, M = 0.5$)

5.3   [Geometric Variance Gamma Process & MEMM] Model

(a) $b_0 = -0.4$ (b) $b_0 = 0$ (c) $b_0 = 0.6$
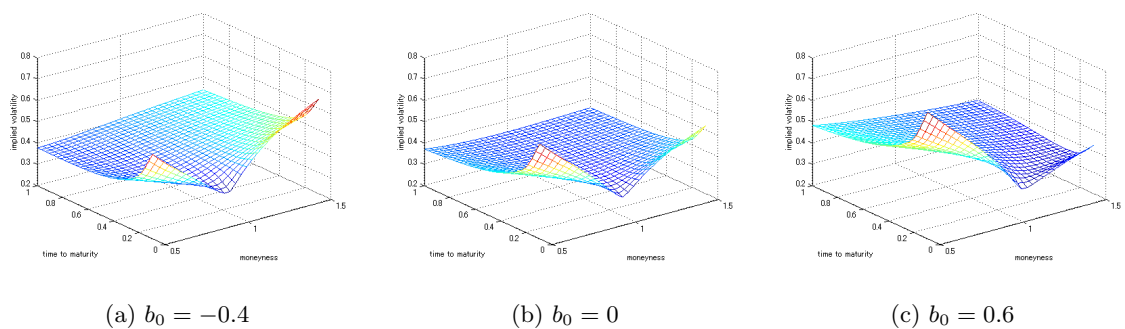
Figure 11: [G-VG Process & MEMM] Model ($C = 5, c_1 = 15, c_2 = 0.5$)

(a) $C = 5$ (b) $C = 10$ (c) $C = 20$
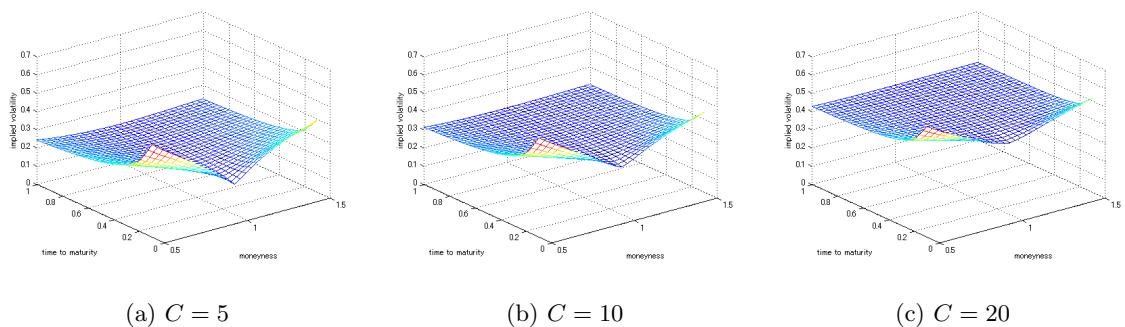
Figure 12: [G-VG Process & MEMM] Model ($b_0 = 0, c_1 = 15, c_2 = 15$)

(a) $c_1 = 15, c_2 = 15$ (b) $c_1 = 28, c_2 = 2$ (c) $c_1 = 10, c_2 = 10$
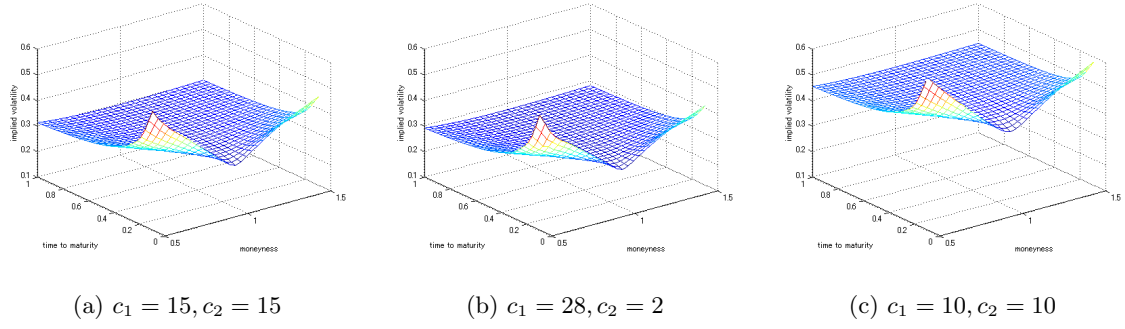
Figure 13: [G-VG Process & MEMM] Model ($b_0 = 0, C = 10$)

6 Implied Volatility Level Calibration of Nikkei225 Index Option by [GLP & MEMM] Models

Let $\{\sigma_j^{(im)}\}$ be the set of implied volatilities obtained in the market, and let $\{\sigma_j^{(im)*}(\Theta)\}$ be the set of the corresponding implied volatilities of the model with the parameter Θ. Then the implied volatility level calibration is to solve the following minimization problem.

$$\min_{\Theta} \sum_j |\sigma_j^{(im)*}(\Theta) - \sigma_j^{(im)}|^2.$$

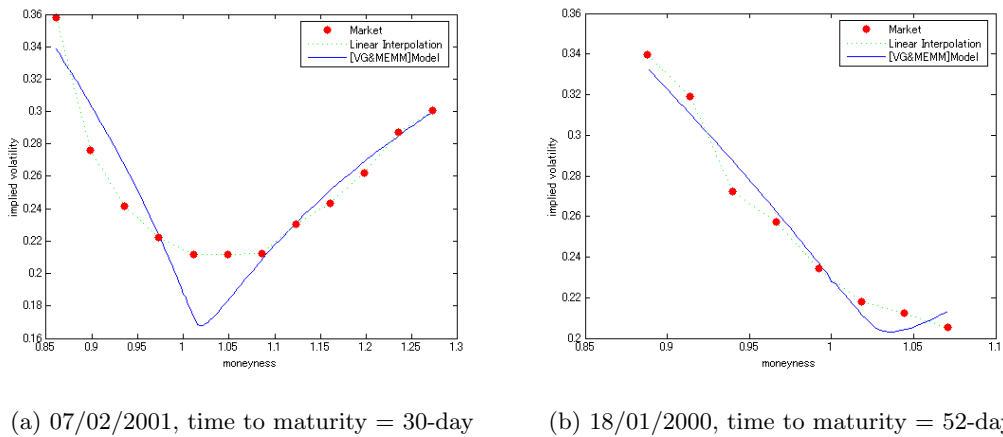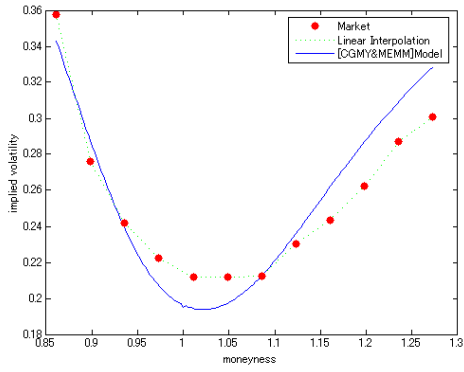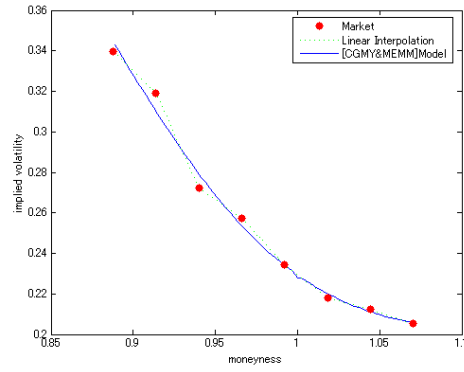(a) 07/02/2001, time to maturity = 30-day (b) 18/01/2000, time to maturity = 52-day

Figure 14: the implied volatility curve of [Geometric VG Process & MEMM] Model and Nikkei 225 call option.
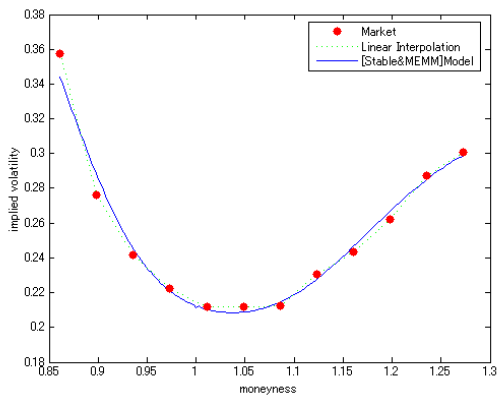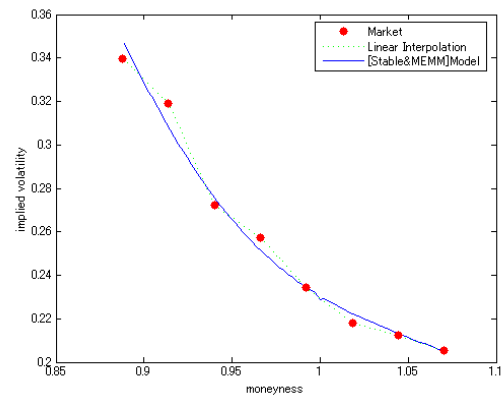
(a) 07/02/2001, time to maturity = 30-day

(b) 18/01/2000, time to maturity = 52-day

Figure 15: the implied volatility curve of [Geometric CGMY Process & MEMM] Model and Nikkei 225 call option.

(a) 07/02/2001, time to maturity = 30-day

(b) 18/01/2000, time to maturity = 52-day

Figure 16: the implied volatility curve of [Geometric Stable Process & MEMM] Model and Nikkei 225 call option.

7   Concluding Remarks

We have only done the implied volatility level calibration. For the selection of suitable model, we have to do the option price level calibration and the fitness analysis. These analyses are the subjects for us to do next.

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