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Mimimal Relative Entropy Martingale Measure of Birth and Death Process

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Abstract

In this article, we investigate the MEMM (Minimal relative Entropy Martingale Measure) of Birth and Death processes and the MEMM of generalized Birth and Death processes. We see that the existence problem of the MEMM is reduced to the problem of solving the corresponding Hamilton-Jacobi-Bellman equation.

1 Introduction

The relative entropy plays very important roles in various fields, for example in the statistical physics, in the information theory, and statistical estimation theory. Recently the ralative entropy has been proved that it is related to the mathematical finance theory. We investigate the MEMM (Minimal relative Entropy Martingale Measure) of Birth and Death processes in this context.

In §2 we formulate our problems as a variation problems. In §3 we introduce the Hamilton-Jacobi-Bellman equation corresponding to the variation problems. In §4 we see that the existence problem of MEMM is reduced to the problem of solving the Hamilton-Jacobi-Bellman equation, and we give an existence theorem of MEMM in §5.

In §6 we extend the above results to the generalized birth and death processes, and we also see an application of our results to the mathematical finance.

Finally in §7 we give some remarks for the studies in the future.

2 Formulation of Problems

We denote by $\mathcal{D}[0,T]$ the space of all *càdlàg* functions, namely, right continuous functions with left limit, and let $\mathcal{F}_t$ be the natural filtrations on the space $\mathcal{D}[0,T]$.

Suppose that a probability P on the measuralbe space $(\mathcal{D}[0,T], \mathcal{F}), \mathcal{F} = \mathcal{F}_T$, is given and that this probability P determines a Birth and Death process $X(t,w) = w(t), w \in \mathcal{D}[0,T]$ such that the jumping measure $\nu(x,dy)$ is

$$\nu(x,dy) = c(x)(p(x)\delta_{\{x+1\}}(dy) + (1-p(x))\delta_{\{x-1\}}(dy)). \qquad (1)$$

We assume here that

$$0 < c_1 \leq c(x) \leq c_2 < \infty, \quad 0 < p_1 \leq p(x) \leq p_2 < 1. \qquad (2)$$

Our purpose is to find the minimal relative entropy martingale measure (MEMM) of the Birth and Death process $w(t), w \in \mathcal{D}[0,T]$.

Now we give the definition of the MEMM. Let $\mathcal{P}$ be the set of all equivalent martingale measures of $w(t)$. (I.e., $\mathcal{P}$ is the set of all Q such that $Q \sim P$ and $w(t)$ is $(Q, \mathcal{F}_t)$ martingale.)

Definition 1 (minimal entropy martingale measure (MEMM)) *If an equivalent martingale measure $\hat{P}$ satisfies the following condition*

$$H(\hat{P}|P) \leq H(Q|P) \qquad \forall Q \in \mathcal{P} \qquad (3)$$

where $H(Q|P)$ is the relative entropy of Q with respect to P, which is given by the following formula

$$H(Q|P) = \begin{cases} E_Q\left[\log\{\frac{dQ}{dP}\}\right], & if \quad Q \ll P, \\ \infty, & otherwise, \end{cases} \qquad (4)$$

then $\hat{P}$ is called the minimal entropy martingale measure (MEMM) of $w(t)$.

The basic properties of MEMM are described in §2 of [3]. For example, it is known that if the MEMM exists then it is unique.

From the results for the case of discrete time processes, or from the results of Nagasawa [7], we can suppose the following conjecture.

Conjecture 1 *If the MEMM $\hat{P}$ exists, then the MEMM is a Markov (i.e., the process $w(t)$ is Markov with respect to $(\hat{P}, \mathcal{F}_t)$).*

If this conjecture is true, then our problem is reduced to the problem to find the probability measure which has the minimal relative entropy tith respect to P in the subset of all Markov equivalent martingale measures. But unfortunately we couldn't have proved the above conjecture yet. So we first put our focus on the Markov equivalent martingale measures, and after that we will investigate the original problem.

Let $\mathcal{M}$ be the set of all Markov equivalent martingale measures. We first investigate the minimal relative entropy Markov martingale measure (i.e. Csiszár's projection on the set $\mathcal{M}$).

From Conjecture 1, we can expect the following conjecture.

Conjecture 2 *Suppose that there is a Csiszár's projection of P, $\tilde{P}$, on $\mathcal{M}$, then $\tilde{P}$ is MEMM (namely, $\tilde{P}$ is the Csiszár's projection on $\mathcal{P}$).*

We will see the relations of our results with the above conjectures later.

By the results of Kunita-Watanabe [2], any Markov equivalent probability measure Q is obtained by the following way:

Set

$$\mathbf{F} = \{f(t,x,y); \mathrm{E}_P[\iint_{[0,T]\times R} |e^{f(t,w(t),y)} - 1|\nu(w(t),dy)dt] < \infty\} \quad (5)$$

and set

$$\alpha(s,t;f,w) = \int_s^t \int f(u, w(u-), y) P(dudy, w) - \int_s^t \int (e^{f(u,w(u),y))} - 1)\nu(w(u), dy)du, \quad (6)$$

where $P(dudy, w)$ means the stochastic integral in the sence of Kunita-Watanabe [2]. Let $Q^{(f)}$ be the probability measure on the space $\mathcal{D}[0,T], \mathcal{F}$ defined by

$$\frac{dQ^{(f)}}{dP} = e^{\alpha(0,T;f,w)} \quad (7)$$

and under the new probability $Q^{(f)}$ $w(t)$ is a jump Markov process with the jumping measure $e^{f(t,x,y)}\nu(x,dy)$.

It is easy to see that the condition of $f(t,x,y)$ such that the coresponding measure $Q^{(f)}$ is a martingale measure of $w(t)$ is

$$\int (y-x)e^{f(t,x,y)}\nu(x,dy) = 0. \quad (8)$$

In the case of Birth and Death process, this condition is equivalent to the condition that $f(t,x,y)$ can be expressed as follows

$$\begin{aligned} f(t,x,y) &= \lambda_0(x)(y-x) + a(t,x), \\ & \qquad y \in supp\,\nu(x,\cdot) = \{x+1, x-1\}, \end{aligned} \tag{9}$$

where

$$\lambda_0(x) = \frac{1}{2}\log\{\frac{1-p(x)}{p(x)}\}, \tag{10}$$

and $a(t,x)$ can be chosen freely under the condition that $f(t,x,y)$ of (8) is in $\mathbf{F}$. We denote by $\mathbf{A}$ the set of all such functions. For $a(t,x)$, define $f_a(t,x,y)$ by

$$f_a(t,x,y) = \begin{cases} \lambda_0(x)(y-x) + a(t,x), & y \in supp\,\nu(x,\cdot) = \{x+1, x-1\} \\ 0, & otherwise, \end{cases} \tag{11}$$

and set $\tilde{\mathbf{F}} = \{f_a \in \mathbf{F}; a \in \mathbf{A}\}$

Now our problem is to find $f_0 \in \tilde{\mathbf{F}}$ such that

$$H(Q^{(f_0)}|P) \le H(Q^{(f_a)}|P), \quad \forall f_a \in \tilde{\mathbf{F}}. \tag{12}$$

3 Variation Problem and Hamilton-Jacobi-Bellman Equation

The problem to obtain the Csiszár's projection on $\mathcal{M}$ is reduced to a variation problem in the following way. As we have seen in the previous section, we have to investigate the following minimization problem

$$\min_{a\in\mathbf{A}} H(Q^{(f_a)}|P) \tag{13}$$

This quantity is equal to

$$\min_{a\in\mathbf{A}} E_P[\alpha(0,T;f_a,w)e^{\alpha(0,T;f_a,w))}]. \tag{14}$$

Thus we know that our probrem is similar to stochastic control problems. So obaying to the idea of control theory, we introduce the Hamilton-Jacobi-Bellman equation.

Set

$$\begin{aligned} \varphi(t,x) &= \min_{a\in\mathbf{A}} H(Q^{f_a}_{(t,x)}|P_{(t,x)}) \\ &= \min_{a\in\mathbf{A}} E_{P_{(t,x)}}[\alpha(t,T;f_a,w)e^{\alpha(t,T;f_a,w)}], \end{aligned} \tag{15}$$

where $P_{(t,x)}$ means conditional probability of P under the condition that $w(t) = x$.

We can carry on the following calculation.

$$\begin{aligned}\varphi(t,x) &= \min_{a\in\mathbf{A}} E_P[\alpha(t,t_1;f_a,w)+\alpha(t_1,T;f_a,w)e^{\alpha(t,T;f_a,w)}] \\ &= \min_{a\in\mathbf{A}}\{E_P[\alpha(t,t_1;f_a,w)e^{\alpha(t,T;f_a,w)}] \\ &\quad +E_P[\alpha(t_1,T;f_a,w)e^{\alpha(t,T;f_a,w)}]\} \\ &= \min_{a\in\mathbf{A}}\{I_1+I_2\} \qquad (16)\end{aligned}$$

$$\begin{aligned}I_1 &= E_P[\alpha(t,t_1;f_a,w)e^{\alpha(t,T;f_a,w)}] \\ &= E_P[E_P[\alpha(t,t_1;f_a,w)e^{\alpha(t,t_1;f_a,w)}e^{\alpha(t_1,T;f_a,w))}|\mathcal{F}_{t_1}]] \\ &= E_P[\alpha(t,t_1;f_a,w)e^{\alpha(t,t_1;f_a,w)}] \\ &= E_P[\iint_{[t,t_1]\times R}\lambda_0(w(s-))(y-w(s-))P(dsdy,w)e^{\alpha(t,t_1;f_a,w)}] \\ &\quad +E_P[\iint_{[t,t_1]\times R}a(s,w(s-))P(dsdy,w)e^{\alpha(t,t_1;f_a,w)}] \\ &\quad -E_p[\int\int_{[t,t_1]\times R}\left(e^{\lambda_0(w(s))(y-w(s))+a(s,w(s))}-1\right)\nu(w(s),dy)ds \\ &\qquad \times e^{\alpha(t,t_1;f_a,w)}] \\ &= I_{1,1}+I_{1,2}+I_{1,3} \qquad (17)\end{aligned}$$

Since $w(s)$ is martingale with respect to the transformed probability measure $e^{\alpha(t,t_1;f_a,w)}dP$, we obtain

$$I_{1,1}=0. \qquad (18)$$

By Theorem 6.2 of Kunita-Watanabe [2], $w(s)$ is a jump Markov process with jumping measure $e^{f_a(s,x,y)}\nu(x,dy)=e^{\lambda_0(x)(y-x)+a(s,x)}\nu(x,dy)$ under the transformed measure $e^{\alpha(t,t_1;f_a,w)}dP$, we obtain

$$\begin{aligned}&\lim_{t_1\downarrow t}\frac{I_{1,2}}{t_1-t} \\ &= a(t,w(t))c(w(t))\left(e^{\lambda_0(w(t))+a(t,w(t))}p(w(t))\right. \\ &\qquad \left.+\,e^{-\lambda_0(w(t))+a(t,w(t))}(1-p(w(t)))\right) \\ &= a(t,w(t))c(w(t))\times 2\sqrt{p(w(s))(1-p(w(s)))}e^{a(t,w(t))} \\ &= a(t,w(t))c(w(t))h(w(t))e^{a(t,w(t))}, \qquad (19)\end{aligned}$$

where we set

$$h(x)=2\sqrt{p(x)(1-p(x))} \qquad (20)$$

for the simplisity of notations.

Next we calculate the term $I_{1,3}$.

$$I_{1,3} = -E_P[\int_t^{t_1} c(w(s))(h(w(s))e^{a(s,w(s))} - 1)ds e^{\alpha(t,t_1;f_a,w)}] \tag{21}$$

Therefore

$$\lim_{t_1 \downarrow t} \frac{I_{1,3}}{t_1 - t} = -c(x)(h(x)e^{a(t,x)} - 1) \tag{22}$$

¿From (18),(19) and (22), we obtain

$$\lim_{t_1 \downarrow t} \frac{I_1}{t_1 - t} = c(x)(h(x)(a(t,x) - 1)e^{a(t,x)} + 1). \tag{23}$$

Next we will investigate the second term I_2 in (16).

$$\begin{aligned} I_2 &= E_P[\alpha(t_1, T; f_a, w)e^{\alpha(t,T;f_a,w)}] \\ &= E_P[E_P[\alpha(t_1, T; f_a, w)e^{\alpha(t,T;f_a,w)}|\mathcal{F}_{t_1}]] \\ &= E_P[e^{\alpha(t,t_1;f_a,w)}E_P[\alpha(t_1, T; f_a, w)e^{\alpha(t_1,T;f_a,w)}|\mathcal{F}_{t_1}]] \end{aligned} \tag{24}$$

We suppose that there is the optimal function $a_0(t,x)$. Then the term I_2 in (16) can be replaced by

$$\begin{aligned} & E_P[e^{\alpha(t,t_1;f_{a_0},w)}E_P[\alpha(t_1, T; f_{a_0}, w)e^{\alpha(t_1,T;f_{a_0},w)}|\mathcal{F}_{t_1}]] \\ =\ & E_P[e^{\alpha(t,t_1;f_{a_0},w)}\varphi(t_1, w(t_1))] \end{aligned} \tag{25}$$

and (15) is equivalent to

$$\varphi(t,x) = \min_{a \in \mathbf{A}}\{I_1 + E_P[e^{\alpha(t,t_1;f_a,w)}\varphi(t_1, w(t_1))]\} \tag{26}$$

By Kunita-Watanabe [2, Theorem 6.2], $\{w(s), t \leq s \leq t_1\}$ is a jump Markov process with the jumping measure $e^{f_a(s,x,y)}\nu(x,dy)$ under the transformed measure $e^{\alpha(t,t_1;f_a,w)}dP(w)$. Therefore we obtain

$$\begin{aligned} & \lim_{t_1 \downarrow t} \frac{E_P[e^{\alpha(t,t_1;f_a,w)}\varphi(t_1, w(t_1))] - \varphi(t,x)}{t_1 - t} \\ =\ & \frac{\partial \varphi}{\partial t}(t,x) + \int (\varphi(t,y) - \varphi(t,x))e^{f_a(t,x,y)}\nu(x,dy) \\ =\ & \frac{\partial \varphi}{\partial t}(t,x) + 2c(x)\sqrt{p(x)(1-p(x))}e^{a(t,x)} \\ & \quad \times (\frac{\varphi(t,x+1) + \varphi(t,x-1)}{2} - \varphi(t,x)) \\ =\ & \frac{\partial \varphi}{\partial t}(t,x) + c(x)h(x)e^{a(t,x)}L\varphi(t,x), \end{aligned} \tag{27}$$

where we set

$$L\varphi(t,x) = (\frac{\varphi(t,x+1) + \varphi(t,x-1)}{2} - \varphi(t,x)) \tag{28}$$

Now we have obtained the optimality equation

$$0 = \min_{a \in \mathbf{A}} \{c(x)(h(x)(a(t,x)-1)e^{a(t,x)}+1) + \tfrac{\partial \varphi}{\partial t}(t,x) + c(x)h(x)e^{a(t,x)}L\varphi(t,x)\}. \tag{29}$$

The equation is the so-called Hamilton-Jacobi-Bellman equation corresponding to the original variation problem (13).

Solving the above minimizing problem, we get

$$a(t,x) = -L\varphi(t,x) \tag{30}$$

and

$$\frac{\partial \varphi}{\partial t}(t,x) = c(x)(h(x)e^{-L\varphi(t,x)} - 1). \tag{31}$$

4 Sufficient Conditions for the Existence of MEMM

In the previous section, we have introduced the variation problem related to the original problem, and we have reduced the problem to the Hamilton-Jacobi-Bellman equation. In this section we will prove that if the Hamilton-Jacobi-Bellman equation has a solution, then there exists the Csiszár's projection P^* of P on $\mathcal{M}$, and that P^* is also the Csiszár's projection on $\mathcal{P}$ (i.e. P^* is the MEMM of Birth and Death process).

Theorem 1 *Supose that the Hamilton-Jacobi-Bellman equation*

$$\frac{\partial \varphi}{\partial t}(t,x) = c(x)(h(x)e^{-L\varphi(t,x)} - 1), \quad 0 \le t \le T, \tag{32}$$

$$\varphi(T,x) = 0 \tag{33}$$

has a bounded solution, and let $a^*(t,x)$ be defined by

$$a^*(t,x) = -L\varphi(t,x). \tag{34}$$

Then the probability measure P^* given by

$$\frac{dP^*}{dP} = e^{\alpha(0,T;f_{a^*},w)} \tag{35}$$

is the MEMM of the Birth and Death process $w(t)$. And it holds true that

$$H(P^*|P) = \varphi(0,a).$$

Proof.
Step 1. From the definition of f_{a^*}, it follows that

$$P^* = Q^{(f_{a^*})} \in \mathcal{M} \subseteq \mathcal{P}. \tag{36}$$

Step 2. We prove that P^* is the MEMM. First we prepare a lemma.

Lemma 1 (Fundamental Lemma, see [5] for example) *Let P, Q, and $\tilde{Q}$ be probability measures defined on a probability space $(\Omega, \mathcal{F})$ such that $P \sim Q \sim \tilde{Q}$. Then it holds that*

$$H(Q|P) \geq \int_\Omega \log[\frac{d\tilde{Q}}{dP}]dQ. \tag{37}$$

Using this lemma, we obtain

$$H(Q|P) \geq E_Q\left[\log\{\frac{dP^*}{dP}\}\right], \quad \forall Q \in \mathcal{P}. \tag{38}$$

We will calculate the right hand side of this inequality.

$$\begin{aligned} \log\{\frac{dP^*}{dP}\} &= \alpha(0,T;f_{a^*},w) \\ &= \iint_{[0,T]\times R} f_{a^*}(s,w(s-),y)P(dsdy,w) \\ &\quad - \iint_{[0,T]\times R}(e^{f_{a^*}(s,w(s),y)}-1)\nu(w(s),y)ds \end{aligned} \tag{39}$$

We mention here that f_{a^*} is expressed in the folowing form

$$\begin{aligned} f_{a^*}(s,x,y) &= (\lambda_0(x) + \frac{\varphi(t,x+1)-\varphi(t,x-1)}{2})(y-x) \\ &\quad -(\varphi(t,y)-\varphi(t,x)), \quad \forall y \in supp\,\nu(x,\cdot). \end{aligned} \tag{40}$$

This fact is easily checked by the use of $a^*(t,x) = -L\varphi(t,x)$ and $supp\,\nu(x,\cdot) = \{x+1, x-1\}$.

Using this equality, we obtain

$$\begin{aligned} \log\{\frac{dP^*}{dP}\} &= \alpha(0,T;f_{a^*},w) \\ &= \iint_{[0,T]\times R}\Big\{(\lambda_0(w(s-)) + \frac{\varphi(s,w(s-)+1)-\varphi(s,w(s-)-1)}{2})(y-w(s-)) \\ &\quad -(\varphi(s,y)-\varphi(s,w(s-)))\Big\}P(dsdy,w) \\ &\quad - \iint_{[0,T]\times R}(e^{f_{a^*}(s,w(s),y)}-1)\nu(w(s),y)ds \end{aligned} \tag{41}$$

The second term of the above formula is calculated as follows.

$$
\begin{aligned}
& \iint_{[0,T]\times R} (e^{f_{a^*}(s,w(s),y)} - 1)\nu(w(s),y)ds \\
= & \iint_{[0,T]\times R} (e^{\lambda_0(w(s))(y-w(s))+a^*(s,w(s))} - 1)\nu(w(s),dy)ds \\
= & \int_0^T c(w(s))(h(w(s))e^{-L(s,w(s))} - 1)ds \\
= & \int_0^T \frac{\partial\varphi}{\partial s}(s,w(s))ds
\end{aligned}
\tag{42}
$$

where the last equality follows from the assumption that $\varphi(t,x)$ is the solution of the Hamilton-Jacobi-Bellman equation (31). Thus we have obtained

$$
\begin{aligned}
& \log\{\frac{dP^*}{dP}\} \\
= & \iint_{[0,T]\times R} \Big\{(\lambda_0(w(s-)) + \frac{\varphi(s,w(s-)+1) - \varphi(s,w(s-)-1)}{2})(y - w(s-)) \\
& -(\varphi(s,y) - \varphi(s,w(s-)))\Big\}P(dsdy,w) - \int_0^T \frac{\partial\varphi}{\partial s}(s,w(s))ds
\end{aligned}
\tag{43}
$$

Since Q is martingale measure and the functions $\lambda_0(x)$ and $\varphi(t,x)$ are bounded by the assumption, it follows that

$$
\begin{aligned}
& E_Q\Big[\iint_{[0,T]\times R}\{\lambda_0(w(s-)) + \tfrac{\varphi(s,w(s-)+1)-\varphi(s,w(s-)-1)}{2}\} \\
& \qquad \times(y-w(s-))P(dsdy,w)\Big] \\
& = 0.
\end{aligned}
\tag{44}
$$

Using this result, form (43) we obtain

$$
\begin{aligned}
& E_Q\left[\log\{\frac{dP^*}{dP}\}\right] \\
= & E_Q\left[\iint_{[0,T]\times R}\{-(\varphi(s,y) - \varphi(s,w(s-)))\}P(dsdy,w) - \int_0^T \frac{\partial\varphi}{\partial s}(s,w(s))ds\right].
\end{aligned}
\tag{45}
$$

On the other hand, by Ito's formula

$$
\begin{aligned}
& \varphi(t,w(t)) - \varphi(0,w(0)) \\
= & \iint_{[0,t]\times R}(\varphi(s,y) - \varphi(s,w(s-)))P(dsdy,w) \\
& + \int_0^t \frac{\partial\varphi}{\partial s}(s,w(s))ds.
\end{aligned}
\tag{46}
$$

From (45) and (46), we obtain the following result

$$E_Q\left[\log\{\frac{dP^*}{dP}\}\right] = \varphi(0,w(0)) - \varphi(T,w(T)) = \varphi(0,w(0)). \tag{47}$$

Therefore we have proved that

$$H(Q|P) \geq E_Q\left[\log\{\frac{dP^*}{dP}\}\right] = \varphi(0,w(0)), \quad \forall Q \in \mathcal{P}. \tag{48}$$

Taking $Q = P^*$ in this formula, we obtain

$$H(P^*|P) = E_{P^*}\left[\log\{\frac{dP^*}{dP}\}\right] = \varphi(0,w(0)), \tag{49}$$

and we have finally obtained

$$H(Q|P) \geq \varphi(0,w(0)) = H(P^*|P), \quad \forall Q \in \mathcal{P}. \tag{50}$$

This proves that P^* is the MEMM, and the equality

$$H(P^*|P) = \varphi(0,a).$$

Proof of the theorem is finished. (Q.E.D.)

Remark 1 *From the results of this Theorem, we know that the process is not necessary temporally homogeneous under the MEMM even if it is temporally homogeneous under the original probability P.*

5 Existence Theorem of MEMM of Birth and Death Process

In this section we prove that the Hamilton-Jacobi-Bellman equation (31) obtained in Section 3 has a solution.

We investigate the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial\varphi}{\partial t}(t,x) = c(x)(h(x)e^{-L\varphi(t,x)} - 1), \quad 0 \leq t \leq T, \tag{51}$$

$$\varphi(T,x) = 0, \tag{52}$$

where L and $h(x)$ are given in Section 3 as follows

$$L\varphi(t,x) = \left(\frac{\varphi(t,x+1) + \varphi(t,x-1)}{2} - \varphi(t,x)\right), \tag{53}$$

and

$$h(x) = 2\sqrt{p(x)(1-p(x)}. \tag{54}$$

Set

$$g(t,x) = e^{-\varphi(t,x)}. \tag{55}$$

Then the equations (51) and (52) are transformed to

$$\begin{aligned}\frac{\partial g}{\partial t}(t,x) &= -g(t,x)c(x)\left(h(x)e^{-L\varphi(t,x)} - 1\right) \\ &= c(x)\left(g(t,x) - h(x)\sqrt{g(t,x+1)g(t,x-1)}\right), \quad (56)\\ & \quad 0 \le t \le T,\end{aligned}$$

and

$$g(T,x) = 1. \tag{57}$$

We set

$$\tau = T - t, \quad and \qquad u(\tau,x) = g(T-\tau,x), \tag{58}$$

then we obtain

$$\begin{aligned}\frac{\partial u}{\partial \tau}(\tau,x) &= -c(x)\left(u(\tau,x) - h(x)\sqrt{u(\tau,x+1)u(\tau,x-1)}\right), (59)\\ & \quad 0 \le \tau \le T,\end{aligned}$$

and

$$u(0,x) = 1. \tag{60}$$

We express this equation in the following form

$$\begin{aligned}& \frac{\partial u}{\partial \tau}(\tau,x) \\ = \ & c(x)h(x)\left[\frac{u(\tau,x+1)+u(\tau,x-1)}{2} - u(\tau,x)\right] - c(x)(1-h(x))u(\tau,x) \\ & +c(x)h(x)\left[\sqrt{u(\tau,x+1)u(\tau,x-1)} - \frac{(u(\tau,x+1)+u(\tau,x-1))}{2}\right], \\ & \quad 0 \le \tau \le T, \qquad (61)\\ & u(0,x) = 1. \qquad (62)\end{aligned}$$

We concider the principal part of this equation

$$\begin{aligned}& \frac{\partial v}{\partial \tau}(\tau,x) \\ = \ & c(x)h(x)\left[\frac{v(\tau,x+1)+v(\tau,x-1)}{2} - v(\tau,x)\right] - c(x)(1-h(x))v(\tau,x), \\ & \quad 0 \le \tau \le T. \qquad (63)\end{aligned}$$

Let $\{q(\tau,x,y)\}$ be the fundamental solutions of the above equation. Then the solution $u(\tau,x)$ of (61) and (62) is given by

$$u(\tau,x) = \sum_y q(\tau,x,y) + \int_0^\tau ds \sum_y q(\tau-s,x,y)c(y)h(y) \times \left[\sqrt{u(s,y+1)u(s,y-1)} - \frac{(u(s,y+1)+u(s,y-1))}{2}\right] \tag{64}$$

$0 < \tau \leq T.$

Here we mention that

$$\sqrt{u(s,y+1)u(s,y-1)} - \frac{(u(s,y+1)+u(s,y-1))}{2} \leq 0 \tag{65}$$

Therefore we obtain

$$u(\tau,x) \leq \sum_y q(\tau,x,y) \leq 1. \tag{66}$$

On the other hand, from the inequality

$$\frac{\partial u}{\partial \tau}(\tau,x) \geq -c(x)u(\tau,x), \tag{67}$$

it follows that

$$u(\tau,x) \geq \exp\{-c(x)\tau\}. \tag{68}$$

¿From (66) and (68) it follows that

$$\exp\{-c(x)\tau\} \leq u(\tau,x) \leq 1, \quad 0 \leq t \leq T. \tag{69}$$

The existence of the solution of the equations (61) and (62) is assured as follows. The right hand side of (64) is Lipshitz continuous functional of $u(\tau,x)$ on the space $C\left([0,T]\times R \to [e^{-C_2 T},1]\right)$. Therefore there exists the unique solution of (64).

Now we have proved the following theorem.

Theorem 2 *Assume that*

$$0 < c_1 \leq c(x) \leq c_2 < \infty, \quad 0 < p_1 \leq p(x) \leq p_2 < 1. \tag{70}$$

Then the Hamilton-Jacobi-Bellman equations (51) and (52) have a unique bounded solution.

Combining Theorem 1 and Theorem 2, we obtain

Theorem 3 *If the assumptions of Theorem 2 are satisfied, then*
(1)The MEMM P^* of Birth and Death process exists, and P^* is defined by (35) in Theorem 1.
(2)Under the MEMM P^*, the process w_t is also a Birth and Death process with the jumping measure

$$\nu_t^*(x,dy) = e^{f_{a^*}(t,x,y)}\nu(x,dy). \tag{71}$$

(3) $H(P^*|P) = \varphi(0,x)$.

Proof. By Theorem 2, the Hamilton-Jacobi-Bellman equation has a solution. Therefore, by Theorem 1, the probability measure P^* defined in Theorem 1 is the MEMM. From the definition of P^*, it is clear that P^* is a Markov probebility. (Q.E.D.)

Remark 2 *We see in (2) of the Theorem that w_t is a Birth and Death process under the MEMM P^*, but it is not necessarily time homogeneous even if it is time homogeneous under the original measure P.*

6 MEMM of Genaralized Birth and Death Processes

In this section we apply the ideas and the methods developed in the previous sections to the generalized birth and death processes. We mean by the generalized birth and death process such a process that is a jump type Markov process with the state space

$$\mathcal{S} = \{a_n, n = 0, \pm 1, \ldots\}, \quad a_{n-1} < a_n < a_{n+1}, n = 0, \pm 1, \ldots, \tag{72}$$

and with the jumping measure

$$\nu(a_n, dy) = c(a_n)(p(a_n)\delta_{\{a_{n+1}\}}(dy) + (1 - p(a_n))\delta_{\{a_{n-1}\}}(dy)). \tag{73}$$

6.1 Formulation

We can suppose, as in section 2, the above process is defined on the probability space $(\mathcal{D}[0,T], \mathcal{F}, P), \mathcal{F} = \mathcal{F}_T$, and that this probability P determines a generalized Birth and Death process $X(t,w) = w(t), w \in \mathcal{D}[0,T]$ such that the jumping measure $\nu(x, dy)$ is given by (73).

We assume as before that

$$0 < c_1 \le c(x) \le c_2 < \infty, \quad 0 < p_1 \le p(x) \le p_2 < 1. \tag{74}$$

We use the same notations as in §2. The condition on $f(t,x,y)$ such that the coresponding measure $Q^{(f)}$ is a martingale measure of $w(t)$ (see (8) in §2) is, in this case, as follows

$$p_n(a_{n+1} - a_n)e^{f(t,a_n,a_{n+1})} + (1 - p_n)(a_{n-1} - a_n)e^{f(t,a_n,a_{n-1})} = 0. \tag{75}$$

It is easy to see that this condition is equivalent to the condition that $f(t,x,y)$ is expressed in the following form

$$\begin{aligned} f(t, a_n, y) &= \lambda_n(y - a_n) + b(t, a_n), \\ & \qquad y \in supp\, \nu(a_n, \cdot) = \{a_{n+1}, a_{n-1}\}, \end{aligned} \tag{76}$$

where

$$\lambda_n = \frac{1}{a_{n+1} - a_{n-1}} \log\{\left(\frac{1 - p(a_n)}{p(a_n)}\right) \left(\frac{a_n - a_{n-1}}{a_{n+1} - a_n}\right)\}, \tag{77}$$

and $b(t, a_n)$ can be chosen freely under the condition that $f(t, x, y)$ of (76) is in $\mathbf{F}$. We denote by $\mathbf{B}$ the set of all such functions. For $b(t, x) \in \mathbf{B}$, define $f_b(t, x, y)$ by

$$f_b(t, a_n, y) = \begin{cases} \lambda_n(y - a_n) + b(t, a_n), & y \in supp\, \nu(a_n, \cdot) = \{a_{n+1}, a_{n-1}\} \\ 0, & otherwise, \end{cases} \tag{78}$$

and set $\tilde{\mathbf{F}} = \{f_b \in \mathbf{F}; b \in \mathbf{B}\}$

Now our problem is to find $f_0 \in \tilde{\mathbf{F}}$ such that

$$H(Q^{(f_0)}|P) \leq H(Q^{(f_b)}|P), \quad \forall f_b \in \tilde{\mathbf{F}}. \tag{79}$$

6.2 Hamilton-Jacobi-Bellman Equation

We investigate the following minimization problem

$$\min_{b \in \mathbf{B}} H(Q^{(f_b)}|P), \tag{80}$$

or eqivalently

$$\min_{b \in \mathbf{B}} E_P[\alpha(0, T; f_b, w) e^{\alpha(0,T;f_b,w))}]. \tag{81}$$

Set

$$\begin{aligned} \varphi(t, x) &= \min_{b \in \mathbf{B}} H(Q^{f_b}_{(t,x)}|P_{(t,x)}) \\ &= \min_{b \in \mathbf{B}} E_{P_{(t,x)}}[\alpha(t, T; f_b, w) e^{\alpha(t,T;f_b,w)}], \end{aligned} \tag{82}$$

where $P_{(t,x)}$ means conditional probability of P under the condition that $w(t) = x$, $x \in \mathcal{S} = \{a_n, n = 0, \pm 1, \ldots\}$.

We can carry on the similar calculations as we have done in §3, and we obtain the Hamilton-Jacobi-Bellman equation corresponding to the new variation problem (80) and (82).

Before we state the Hamilton-Jacobi-Bellman equation, we have to prepare several notations. We set

$$\mu_n = p_n^{\left(\frac{a_n - a_{n-1}}{a_{n+1} - a_{n-1}}\right)} (1 - p_n)^{\left(\frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}}\right)}, \tag{83}$$

$$l_n^{(+)} = \left(\frac{a_n - a_{n-1}}{a_{n+1} - a_n}\right)^{\left(\frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}}\right)}, \tag{84}$$

$$l_n^{(-)} = \left(\frac{a_n - a_{n-1}}{a_{n+1} - a_n}\right)^{\left(\frac{a_{n-1}-a_n}{a_{n+1}-a_{n-1}}\right)}, \tag{85}$$

$$h_n^{(+)} = e^{\lambda_n(a_{n+1}-a_n)} p_n = \mu_n l_n^{(+)}, \tag{86}$$

$$h_n^{(-)} = e^{\lambda_n(a_{n-1}-a_n)}(1 - p_n) = \mu_n l_n^{(-)}, \tag{87}$$

$$h_n = h_n^{(+)} + h_n^{(-)} = \mu_n(l_n^{(+)} + l_n^{(-)}), \tag{88}$$

and we introduce the following operators

$$\begin{aligned} L^{(+)}\varphi(t, a_n) &= (\varphi(t, a_{n+1}) - \varphi(t, a_n)), & (89)\\ L^{(-)}\varphi(t, a_n) &= (\varphi(t, a_{n-1}) - \varphi(t, a_n)), & (90) \end{aligned}$$

$$\tilde{L}\varphi(t, a_n) = \left(\frac{l_n^{(+)}}{l_n^{(+)} + l_n^{(-)}}\right) L^{(+)}\varphi(t, a_n) + \left(\frac{l_n^{(-)}}{l_n^{(+)} + l_n^{(-)}}\right) L^{(-)}\varphi(t, a_n). \tag{91}$$

Using these notations, we can state the Hamilton-Jacobi-Bellman equation as

$$\begin{aligned} 0 = \min_{b \in \mathbf{B}} \quad & \{c(a_n)(h_n(b(t, a_n) - 1)e^{b(t,a_n)} + 1) + \tfrac{\partial \varphi}{\partial t}(t, a_n) \\ & + c(a_n) h_n e^{b(t,a_n)} \tilde{L}\varphi(t, a_n)\}. \end{aligned} \tag{92}$$

Solving the above minimizing problem, we get

$$b(t, a_n) = -\tilde{L}\varphi(t, a_n) \tag{93}$$

and, using this fact and (92) again, we obtain

$$\frac{\partial \varphi}{\partial t}(t, a_n) = c(a_n)(h_n e^{-\tilde{L}\varphi(t,a_n)} - 1). \tag{94}$$

If the equation

$$\frac{\partial \varphi}{\partial t}(t, a_n) = c(a_n)(h_n e^{-\tilde{L}\varphi(t,a_n)} - 1), \tag{95}$$

$$\varphi(T, a_n) = 0, \tag{96}$$

has a solution and if the solution has good properties, then we can do the similar argument with those which we have done in Theorem 1 of §4 and its proof. (But, unfortunately, we can not do the exactly same argument. We see these situations in the later, after we study the existence probrem of H-J-B equation.

6.3 Existence of Solution of H-J-B Equation

We next investigate the Hamilton-Jacobi-Bellman equation obtained in the previous subsection. first aim is to get sufficient conditions for the existence of the solution.

We investigate the following Hamilton-Jacobi-Bellman equation

$$\frac{\partial \varphi}{\partial t}(t, a_n) = c(a_n)(h_n e^{-\tilde{L}\varphi(t,a_n)} - 1), \tag{97}$$

$$\varphi(T, a_n) = 0, \tag{98}$$

where

$$h_n = h_n^{(+)} + h_n^{(-)} = \mu_n(l_n^{(+)} + l_n^{(-)}), \tag{99}$$

$$\tilde{L}\varphi(t, a_n) = \left(\frac{l_n^{(+)}}{l_n^{(+)} + l_n^{(-)}}\right) L^{(+)}\varphi(t, a_n) + \left(\frac{l_n^{(-)}}{l_n^{(+)} + l_n^{(-)}}\right) L^{(-)}\varphi(t, a_n). \tag{100}$$

Set

$$g(t, x) = e^{-\varphi(t,x)}. \tag{101}$$

Then the equations (97) and (98) are transformed to

$$\begin{aligned} \frac{\partial g}{\partial t}(t, a_n) &= -g(t, a_n)c(a_n)\left(h_n e^{-\tilde{L}\varphi(t,a_n)} - 1\right) \\ &= c(a_n)\left(g(t, a_n) - h_n\left(g(t, a_{n+1})^{\rho_n} g(t, a_{n-1})^{(1-\rho_n)}\right)\right), \\ & \quad 0 \le t \le T, \end{aligned} \tag{102}$$

$$g(T, x) = 1, \tag{103}$$

where we set

$$\rho_n = \frac{l_n^{(+)}}{l_n^{(+)} + l_n^{(-)}}. \tag{104}$$

We set

$$\tau = T - t, \quad and \qquad u(\tau, x) = g(T - \tau, x), \tag{105}$$

then we obtain

$$\begin{aligned} \frac{\partial u}{\partial \tau}(\tau, a_n x) &= c(a_n)\left(u(\tau, a_n) - h_n\left(u(\tau, a_{n+1})^{\rho_n} u(\tau, a_{n-1})^{(1-\rho_n)}\right)\right), \\ & \quad 0 \le \tau \le T, \end{aligned} \tag{106}$$

and

$$u(0, x) \equiv 1. \tag{107}$$

We express this equation in the following form

$$
\begin{aligned}
&\frac{\partial u}{\partial \tau}(\tau, a_n) \\
= \;& c(a_n)h_n\left[(\rho_n u(\tau, a_{n+1}) + (1-\rho_n)u(\tau, a_{n-1})) - u(\tau, a_n)\right] - c(a_n)(1-h_n))u(\tau, a_n x) \\
& + c(a_n)h_n\left[u(\tau, a_{n+1})^{\rho_n} u(\tau, a_{n-1})^{(1-\rho_n)} - (\rho_n u(\tau, a_{n+1}) + (1-\rho_n)u(\tau, a_{n-1}))\right], \\
& \quad 0 \le \tau \le T, \qquad (108) \\
& u(0, x) = 1. \qquad (109)
\end{aligned}
$$

We concider the principal part of this equation

$$
\begin{aligned}
&\frac{\partial v}{\partial \tau}(\tau, a_n) \\
= \;& c(a_n)h_n\left[(\rho_n v(\tau, a_{n+1}) + (1-\rho_n)v(\tau, a_{n-1})) - v(\tau, a_n)\right] - c(a_n)(1-h_n))v(\tau, a_n), \\
& \quad 0 \le \tau \le T. \qquad (110)
\end{aligned}
$$

This linear equation has a solution. Let $\{q(\tau, x, y)\}$ be the fundamental solutions of the above equation. Then the solution $u(\tau, x)$ of (108) and (109) is given by

$$
\begin{aligned}
u(\tau, a_n) \;=\; & \sum_y q(\tau, a_n, y) + \int_0^\tau ds \sum_y q(\tau - s, a_n, y)c(a_n)h_n \\
& \times \left[u(s, a_{n+1})^{\rho_n} u(s, a_{n-1})^{(1-\rho_n)} - (\rho_n u(s, a_{n+1}) + (1-\rho_n)u(s, a_{n-1}))\right], \\
& \quad 0 < \tau \le T. \qquad (111)
\end{aligned}
$$

Here we mention that

$$
u(s, a_{n+1})^{\rho_n} u(s, a_{n-1})^{(1-\rho_n)} - (\rho_n u(s, a_{n+1}) + (1-\rho_n)u(s, a_{n-1})) \le 0 \qquad (112)
$$

Therefore we obtain

$$
u(\tau, x) \le \sum_{y \in \mathcal{S}} q(\tau, x, y) \le 1. \qquad (113)
$$

On the other hand, from the inequality

$$
\frac{\partial u}{\partial \tau}(\tau, x) \ge -c(x)u(\tau, x), \qquad (114)
$$

it follows that

$$
u(\tau, x) \ge \exp\{-c(x)\tau\}. \qquad (115)
$$

From (113) and (115) it follows that

$$
\exp\{-c(x)\tau\} \le u(\tau, x) \le 1, \quad 0 \le \tau \le T. \qquad (116)
$$

Now we can prove the following theorem in the same way as in §5.

Theorem 4 *Assume that*

$$0 < c_1 \le c(x) \le c_2 < \infty, \quad 0 < p_1 \le p(x) \le p_2 < 1. \tag{117}$$

Then the Hamilton-Jacobi-Bellman equations (97) and (98) have a unique bounded solution.

6.4 Existence of MEMM

Let $\varphi(t, a_n)$ be the solution of H-J-B equation (97) and (98), and set

$$b^*(t, a_n) = -\tilde{L}\varphi(t, a_n). \tag{118}$$

Then the function

$$f_{b^*}(t, a_n, y) = \begin{cases} \lambda_n(y - a_n) + b^*(t, a_n), & y \in supp\, \nu(a_n, \cdot) = \{a_{n+1}, a_{n-1}\} \\ 0, & otherwise, \end{cases} \tag{119}$$

is well-defined, and we define the new probability measure P^* by

$$\frac{dP^*}{dP} = e^{\alpha(0,T; f_{b^*}, w)}. \tag{120}$$

Here we assume that P^* is well-defined.

It is easy to see that this function $f_{b^*}(t, a_n, y)$ can be expressed in the following form

$$f_{b^*}(t, a_n, y) = \left\{ \lambda_n + \frac{1}{a_{n+1} - a_{n-1}} (\varphi(t, a_{n+1}) - \varphi(t, a_{n-1})) \right\} (y - a_n) - (\varphi(t, y) - \varphi(t, a_n))$$

$$= \frac{1}{a_{n+1} - a_{n-1}} \left(\log \left\{ \left(\frac{1 - p_n}{p_n} \right) \left(\frac{a_n - a_{n-1}}{a_{n+1} - a_n} \right) \right\} + (\varphi(t, a_{n+1}) - \varphi(t, a_{n-1})) \right) (y - a_n)$$
$$- (\varphi(t, y) - \varphi(t, a_n)), \tag{121}$$
$$y \in supp\, \nu(a_n, \cdot) = \{a_{n+1}, a_{n-1}\}.$$

If the term $\frac{1}{a_{n+1} - a_{n-1}}$ is bounded, then the arguments we have done in §4 are available and we could obtain the similar results as Theorem 1 (in §4) and Theorem 3 (in §5). However, the boundedness of $\frac{1}{a_{n+1} - a_{n-1}}$ is not true in the general cases. So we need new ideas and methods to proceed the discussion.

We see only the special case such that the price process is the geometric birth and death proces in the next subsection.

6.5 Application to Finance Theory

In this subsection we investigate the existence theorem of the MEMM for the geometric birth and death proces, namely for the case

$$a_n = e^n, \quad n = 0, \pm 1, \dots . \tag{122}$$

Then the function $f_{b^*}(t, a_n, y)$ of (121) is

$$f_{b^*}(t, e^n, y) = \frac{e}{e^2 - 1}\frac{1}{e^n}\left(\log\left\{\left(\frac{1-p_n}{p_n}\right)e^{-1}\right\} + (\varphi(t, e^{n+1}) - \varphi(t, e^{n-1}))\right)(y - e^n)$$
$$-(\varphi(t, y) - \varphi(t, e^n)), \tag{123}$$
$$y \in supp\,\nu(a_n, \cdot) = \{e^{n+1}, e^{n-1}\}.$$

In this case we can apply the technique of [6], and we obtain

Theorem 5 *If the assumptions of Theorem 4 are satisfied, then the MEMM P^* of the geometric Birth and Death process exists, and it has the following properties,*
(1)The MEMM P^* is defined by (120).
(2)Under the MEMM P^*, the process w_t is also a geometric Birth and Death process with the jumping measure

$$\nu_t^*(x, dy) = e^{f_{b^*}(t,x,y)}\nu(x, dy). \tag{124}$$

(3) $H(P^*|P) = \varphi(0, x)$.

Proof. By Theorem 4, the Hamilton-Jacobi-Bellman equation has a bounded solution. We can see that $f_{b^*}(t, a_n, y) \in \mathbf{F}$ and that the probability measure P^* is well-defined by (120). We can follow the story in §4 except the part of the proof of the fact that P^* is MEMM. For the proof of this fact we can adopt the discrete time approximation method which is developed in the proof of Theorem 1 of [6]. (Q.E.D.)

7 Concluding Remarks

1) In this paper, we have investigated only such a case that the pricing process is Birth and Death process. We would like to investigate the cases where the price processes are more general Markov processes.
2) Secondarily we would like to discuss the same problems for the semi-martingale price processes.

References

[1] Chan, T. (1999), "Pricing Contingent Claims on Stocks Driven by Lévy Processes", *The Annals of Applied Probability* 9, No. 2, 504-528.

[2] Kunita, H. and Watanabe, S. (1967), "On Square-integrable Martingales," *Nagoya Math. J.* 30, 209-245.

[3] Y. Miyahara (1996), " Canonica Martingale Measures of Incomplete Assets Markets" in " Probability Theory and Mathematical Statistics: Proceedings of the Seventh Japan-Russia Symposium, Tokyo 1995 (eds. S. Watanabe et al)," pp.343-352.

[4] Y. Miyahara (1996), " Canonical Martingale Measures and Minimal Martingale Measures of Incomplete Assets Markets," The Australian National University Reseach Report, No. FMRR 007-96(pp. 95-100).

[5] Miyahara, Y.(1999), " Minimal Entropy Martingale Measures of Jump Type Price Processes in Incomplete Assets Markets," *Asian-Pacific Financial Markets*, Vol. 6, No. 2, pp. 97-113.

[6] Miyahara, Y.(2000), "Minimal Relative Entropy Martingale Measures of Geometric Lévy Processes and its Applications to Option Pricing Theory," (revised version of "Minimal Relative Entropy Martingale Measures of Geometric Lévy Processes and Option Pricing Models in Incomplete Markets," *Discussion Papers in Economics, Nagoya City University,* No. 249(1999), pp. 1-8.)

[7] Nagasawa, M.(1993), *Schrödinger Equations and Diffusion Theory*, Birkhäuser Verlag.